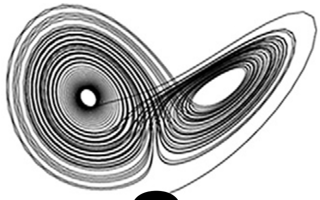


2
Planar Linear Systems

In this chapter we begin the study of *systems of differential equations.* A system of differential equations is a collection of n interrelated differential equations of the form

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, \dots, x_n). \end{aligned}$$

Here the functions f_j are real-valued functions of the $n+1$ variables $x_1, x_2, \dots,$ x_n, and t. Unless otherwise specified, we will always assume that the f_j are C^∞ functions. This means that the partial derivatives of all orders of the f_j exist and are continuous.

To simplify notation, we will use vector notation:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We often write the vector X as $(x_1, \dots, x_n)$ to save space.

Our system may then be written more concisely as

$$X' = F(t, X)$$

where

$$F(t, X) = \begin{pmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{pmatrix}.$$

A solution of this system is then a function of the form $X(t) = (x_1(t), \ldots, x_n(t))$ that satisfies the equation, so that

$$X'(t) = F(t, X(t))$$

where $X'(t) = (x_1'(t), \ldots, x_n'(t))$. Of course, at this stage, we have no guarantee that there is such a solution, but we will begin to discuss this complicated question in Section 2.7.

The system of equations is called *autonomous* if none of the f_j depends on t, so the system becomes $X' = F(X)$. For most of the rest of this book we will be concerned with autonomous systems.

In analogy with first-order differential equations, a vector X_0 for which $F(X_0) = 0$ is called an *equilibrium point* for the system. An equilibrium point corresponds to a constant solution $X(t) \equiv X_0$ of the system as before.

Just to set some notation once and for all, we will always denote real variables by lowercase letters such as x, y, x_1, x_2, t, and so forth. Real-valued functions will also be written in lowercase such as $f(x, y)$ or $f_1(x_1, \ldots, x_n, t)$. We will reserve capital letters for vectors such as $X = (x_1, \ldots, x_n)$, or for vector-valued functions such as

$$F(x, y) = (f(x, y), g(x, y))$$

or

$$H(x_1, \ldots, x_n) = \begin{pmatrix} h_1(x_1, \ldots, x_n) \\ \vdots \\ h_n(x_1, \ldots, x_n) \end{pmatrix}.$$

We will denote n-dimensional Euclidean space by $\mathbb{R}^n$, so that $\mathbb{R}^n$ consists of all vectors of the form $X = (x_1, \ldots, x_n)$.

2.1 Second-Order Differential Equations

Many of the most important differential equations encountered in science and engineering are second-order differential equations. These are differential equations of the form

$$x'' = f(t, x, x').$$

Important examples of second-order equations include Newton's equation

$$mx'' = f(x),$$

the equation for an RLC circuit in electrical engineering

$$LCx'' + RCx' + x = v(t),$$

and the mainstay of most elementary differential equations courses, the forced harmonic oscillator

$$mx'' + bx' + kx = f(t).$$

We will discuss these and more complicated relatives of these equations at length as we go along. First, however, we note that these equations are a special subclass of two-dimensional systems of differential equations that are defined by simply introducing a second variable $y = x'$.

For example, consider a second-order constant coefficient equation of the form

$$x'' + ax' + bx = 0.$$

If we let $y = x'$, then we may rewrite this equation as a system of first-order equations

$$\begin{aligned} x' &= y \\ y' &= -bx - ay. \end{aligned}$$

Any second-order equation can be handled in a similar manner. Thus, for the remainder of this book, we will deal primarily with systems of equations.

2.2 Planar Systems

For the remainder of this chapter we will deal with autonomous systems in $\mathbb{R}^2$, which we will write in the form

$$x' = f(x, y)$$
$$y' = g(x, y)$$

thus eliminating the annoying subscripts on the functions and variables. As above, we often use the abbreviated notation $X' = F(X)$ where $X = (x, y)$ and $F(X) = F(x, y) = (f(x, y), g(x, y))$.

In analogy with the slope fields of Chapter 1, we regard the right-hand side of this equation as defining a *vector field* on $\mathbb{R}^2$. That is, we think of $F(x, y)$ as representing a vector whose x- and y-components are $f(x, y)$ and $g(x, y)$, respectively. We visualize this vector as being based at the point (x, y). For example, the vector field associated to the system

$$x' = y$$
$$y' = -x$$

is displayed in Figure 2.1. Note that, in this case, many of the vectors overlap, making the pattern difficult to visualize. For this reason, we always draw a *direction field* instead, which consists of scaled versions of the vectors.

A solution of this system should now be thought of as a parameterized curve in the plane of the form $(x(t), y(t))$ such that, for each t, the tangent vector at the point $(x(t), y(t))$ is $F(x(t), y(t))$. That is, the solution curve $(x(t), y(t))$ winds its way through the plane always tangent to the given vector $F(x(t), y(t))$ based at $(x(t), y(t))$.

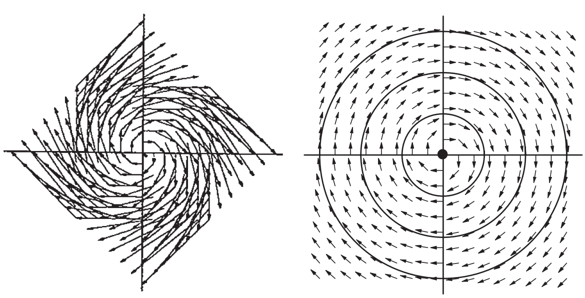


Figure 2.1 The vector field, direction field, and several solutions for the system $x' = y$, $y' = -x$.

Example. The curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a \ \sin t \\ a \ \cos t \end{pmatrix}$$

for any $a \in \mathbb{R}$ is a solution of the system

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

since

$$\begin{aligned} x'(t) &= a \cos t = y(t) \\ y'(t) &= -a \sin t = -x(t) \end{aligned}$$

as required by the differential equation. These curves define circles of radius $|a|$ in the plane that are traversed in the clockwise direction as t increases. When $a = 0$, the solutions are the constant functions $x(t) \equiv 0 \equiv y(t)$. ■

Note that this example is equivalent to the second-order differential equation $x'' = -x$ by simply introducing the second variable $y = x'$. This is an example of a *linear* second-order differential equation, which, in more general form, can be written

$$a(t)x'' + b(t)x' + c(t)x = f(t).$$

An important special case of this is the linear, *constant coefficient* equation

$$ax'' + bx' + cx = f(t),$$

which we write as a system as

$$\begin{aligned} x' &= y \\ y' &= -\frac{c}{a}x - \frac{b}{a}y + \frac{f(t)}{a}. \end{aligned}$$

An even more special case is the *homogeneous* equation in which $f(t) \equiv 0$.

Example. One of the simplest yet most important second-order, linear, constant coefficient differential equations is the equation for a *harmonic oscillator*. This equation models the motion of a mass attached to a spring. The spring is attached to a vertical wall and the mass is allowed to slide along a horizontal track. We let x denote the displacement of the mass from its natural resting place (with $x > 0$ if the spring is stretched and $x < 0$ if the spring is compressed). Therefore the velocity of the moving mass is $x'(t)$ and the acceleration is $x''(t)$. The spring exerts a restorative force proportional to $x(t)$. In addition there is a frictional force proportional to $x'(t)$ in the direction opposite to that of the motion. There are three parameters for this system: m denotes the mass of the oscillator, $b \geq 0$ is the *damping constant*, and $k > 0$ is the *spring constant*. Newton's law states that the force acting on the oscillator is equal to mass times acceleration. Therefore the differential equation for the damped harmonic oscillator is

$$mx'' + bx' + kx = 0.$$

If $b = 0$, the oscillator is said to be *undamped*; otherwise, we have a *damped* harmonic oscillator. This is an example of a second-order, linear, constant coefficient, homogeneous differential equation. As a system, the harmonic oscillator equation becomes

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x - \frac{b}{m}y. \end{aligned}$$

More generally, the motion of the mass-spring system can be subjected to an external force (such as moving the vertical wall back and forth periodically). Such an external force usually depends only on time, not position, so we have a more general forced harmonic oscillator system

$$mx'' + bx' + kx = f(t)$$

where $f(t)$ represents the external force. This is now a nonautonomous, second-order, linear equation. ■

2.3 Preliminaries from Algebra

Before proceeding further with systems of differential equations, we need to recall some elementary facts regarding systems of algebraic equations. We will often encounter simultaneous equations of the form

$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}$$

where the values of a, b, c, and d as well as α and β are given. In matrix form, we may write this equation as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

We denote by A the 2×2 coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This system of equations is easy to solve, assuming that there is a solution. There is a unique solution of these equations if and only if the *determinant* of A is nonzero. Recall that this determinant is the quantity given by

$$\det A = ad - bc.$$

If $\det A = 0$, we may or may not have solutions, but if there is a solution, then in fact there must be infinitely many solutions.

In the special case where $\alpha = \beta = 0$, we always have infinitely many solutions of

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

when $\det A = 0$. Indeed, if the coefficient a of A is nonzero, we have $x = -(b/a)y$ and so

$$-c\left(\frac{b}{a}\right)y + dy = 0.$$

Thus $(ad - bc)y = 0$. Since $\det A = 0$, the solutions of the equation assume the form $(-(b/a)y, y)$ where y is arbitrary. This says that every solution lies on a straight line through the origin in the plane. A similar line of solutions occurs as long as at least one of the entries of A is nonzero. We will not worry too much about the case where all entries of A are 0; in fact, we will completely ignore it.

Let V and W be vectors in the plane. We say that V and W are *linearly independent* if V and W do not lie along the same straight line through the origin. The vectors V and W are *linearly dependent* if either V or W is the zero vector or if both lie on the same line through the origin.

A geometric criterion for two vectors in the plane to be linearly independent is that they do not point in the same or opposite directions. That is, V and W

are linearly independent if and only if $V \neq \lambda W$ for any real number λ. An equivalent algebraic criterion for linear independence is as follows:

Proposition. *Suppose $V = (v_1, v_2)$ and $W = (w_1, w_2)$. Then V and W are linearly independent if and only if*

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0.$$

For a proof, see Exercise 11 at the end of this chapter. ■

Whenever we have a pair of linearly independent vectors V and W, we may always write any vector $Z \in \mathbb{R}^2$ in a unique way as a *linear combination* of V and W. That is, we may always find a pair of real numbers α and β such that

$$Z = \alpha V + \beta W.$$

Moreover, α and β are unique. To see this, suppose $Z = (z_1, z_2)$. Then we must solve the equations

$$z_1 = \alpha v_1 + \beta w_1$$
$$z_2 = \alpha v_2 + \beta w_2$$

where the v_i, w_i, and z_i are known. But this system has a unique solution (α, β) since

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0.$$

The linearly independent vectors V and W are said to define a *basis* for $\mathbb{R}^2$. Any vector Z has unique "coordinates" relative to V and W. These coordinates are the pair (α, β) for which $Z = \alpha V + \beta W$.

Example. The unit vectors $E_1 = (1, 0)$ and $E_2 = (0, 1)$ obviously form a basis called the *standard basis* of $\mathbb{R}^2$. The coordinates of Z in this basis are just the "usual" Cartesian coordinates (x, y) of Z. ■

Example. The vectors $V_1 = (1, 1)$ and $V_2 = (-1, 1)$ also form a basis of $\mathbb{R}^2$. Relative to this basis, the coordinates of E_1 are $(1/2, -1/2)$ and those of E_2 are

$(1/2, 1/2)$, because

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

These "changes of coordinates" will become important later. ■

Example. The vectors $V_1 = (1, 1)$ and $V_2 = (-1, -1)$ do not form a basis of $\mathbb{R}^2$ since these vectors are collinear. Any linear combination of these vectors is of the form

$$\alpha V_1 + \beta V_2 = \begin{pmatrix} \alpha - \beta \\ \alpha - \beta \end{pmatrix},$$

which yields only vectors on the straight line through the origin, V_1, and V_2. ■

2.4 Planar Linear Systems

We now further restrict our attention to the most important class of planar systems of differential equations, namely, linear systems. In the autonomous case, these systems assume the simple form

$$x' = ax + by$$
$$y' = cx + dy$$

where a, b, c, and d are constants. We may abbreviate this system by using the *coefficient matrix* A where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the linear system may be written as

$$X' = AX.$$

Note that the origin is always an equilibrium point for a linear system. To find other equilibria, we must solve the linear system of algebraic equations

$$ax + by = 0$$
$$cx + dy = 0.$$

This system has a nonzero solution if and only if det $A = 0$. As we saw previously, if det $A = 0$, then there is a straight line through the origin on which each point is an equilibrium. Thus we have:

Proposition. *The planar linear system $X' = AX$ has*

1. *A unique equilibrium point* $(0, 0)$ *if det* $A \neq 0$.
2. *A straight line of equilibrium points if det $A = 0$ (and A is not the* 0 *matrix).* ■

2.5 Eigenvalues and Eigenvectors

Now we turn to the question of finding nonequilibrium solutions of the linear system $X' = AX$. The key observation here is this: Suppose V_0 is a nonzero vector for which we have $AV_0 = \lambda V_0$ where $\lambda \in \mathbb{R}$. Then the function

$$X(t) = e^{\lambda t} V_0$$

is a solution of the system. To see this, we compute

$$\begin{aligned} X'(t) &= \lambda e^{\lambda t} V_0 \\ &= e^{\lambda t}(\lambda V_0) \\ &= e^{\lambda t}(AV_0) \\ &= A(e^{\lambda t} V_0) \\ &= AX(t) \end{aligned}$$

so $X(t)$ does indeed solve the system of equations. Such a vector V_0 and its associated scalar have names:

Definition

A nonzero vector V_0 is called an *eigenvector* of A if $AV_0 = \lambda V_0$ for some λ. The constant λ is called an *eigenvalue* of A.

As we observed, there is an important relationship between eigenvalues, eigenvectors, and solutions of systems of differential equations:

Theorem. *Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ. Then the function $X(t) = e^{\lambda t} V_0$ is a solution of the system $X' = AX$.* ■

Note that if V_0 is an eigenvector for A with eigenvalue λ, then any nonzero scalar multiple of V_0 is also an eigenvector for A with eigenvalue λ. Indeed, if $AV_0 = \lambda V_0$, then

$$A(\alpha V_0) = \alpha A V_0 = \lambda(\alpha V_0)$$

for any nonzero constant α.

Example. Consider

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

Then A has an eigenvector $V_0 = (3, 1)$ with associated eigenvalue $\lambda = 2$ since

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}. \qquad \blacksquare$$

Similarly, $V_1 = (1, -1)$ is an eigenvector with associated eigenvalue $\lambda = -2$.

Thus, for the system

$$X' = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} X$$

we now know three solutions: the equilibrium solution at the origin together with

$$X_1(t) = e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad X_2(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We will see that we can use these solutions to generate *all* solutions of this system in a moment, but first we address the question of how to find eigenvectors and eigenvalues.

To produce an eigenvector $V = (x, y)$, we must find a nonzero solution (x, y) of the equation

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that there are three unknowns in this system of equations: the two components of V as well as λ. Let I denote the 2×2 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we may rewrite the equation in the form

$$(A - \lambda I)V = 0,$$

where 0 denotes the vector $(0, 0)$.

Now $A - \lambda I$ is just a 2×2 matrix (having entries involving the variable λ), so this linear system of equations has nonzero solutions if and only if $\det(A - \lambda I) = 0$, as we saw previously. But this equation is just a quadratic equation in λ, whose roots are therefore easy to find. This equation will appear over and over in the sequel; it is called the *characteristic equation*. As a function of λ, we call $\det(A - \lambda I)$ the *characteristic polynomial.* Thus the strategy to generate eigenvectors is first to find the roots of the characteristic equation. This yields the eigenvalues. Then we use each of these eigenvalues to generate in turn an associated eigenvector.

Example. We return to the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

We have

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 3 \\ 1 & -1 - \lambda \end{pmatrix}.$$

So the characteristic equation is

$$\det(A - \lambda I) = (1 - \lambda)(-1 - \lambda) - 3 = 0.$$

Simplifying, we find

$$\lambda^2 - 4 = 0,$$

which yields the two eigenvalues $\lambda = \pm 2$. Then, for $\lambda = 2$, we next solve the equation

$$(A - 2I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In component form, this reduces to the system of equations

$$\begin{aligned} (1 - 2)x + 3y &= 0 \\ x + (-1 - 2)y &= 0 \end{aligned}$$

or $-x + 3y = 0$, because these equations are redundant. Thus any vector of the form $(3y, y)$ with $y \neq 0$ is an eigenvector associated to $\lambda = 2$. In similar fashion, any vector of the form $(y, -y)$ with $y \neq 0$ is an eigenvector associated to $\lambda = -2$. ■

Of course, the astute reader will notice that there is more to the story of eigenvalues, eigenvectors, and solutions of differential equations than what we have described previously. For example, the roots of the characteristic equation may be complex, or they may be repeated real numbers. We will handle all of these cases shortly, but first we return to the problem of solving linear systems.

2.6 Solving Linear Systems

As we saw in the example in the previous section, if we find two real roots λ_1 and λ_2 (with $\lambda_1 \neq \lambda_2$) of the characteristic equation, then we may generate a pair of solutions of the system of differential equations of the form $X_i(t) = e^{\lambda_i t} V_i$ where V_i is the eigenvector associated to λ_i. Note that each of these solutions is a *straight-line solution.* Indeed, we have $X_i(0) = V_i$, which is a nonzero point in the plane. For each t, $e^{\lambda_i t} V_i$ is a scalar multiple of V_i and so lies on the straight ray emanating from the origin and passing through V_i. Note that, if $\lambda_i > 0$, then

$$\lim_{t \to \infty} |X_i(t)| = \infty$$

and

$$\lim_{t \to -\infty} X_i(t) = (0, 0).$$

The magnitude of the solution $X_i(t)$ increases monotonically to ∞ along the ray through V_i as t increases, and $X_i(t)$ tends to the origin along this ray in backward time. The exact opposite situation occurs if $\lambda_i < 0$, whereas, if $\lambda_i = 0$, the solution $X_i(t)$ is the constant solution $X_i(t) = V_i$ for all t.

So how do we find all solutions of the system given this pair of special solutions? The answer is now easy and important. Suppose we have two distinct real eigenvalues λ_1 and λ_2 with eigenvectors V_1 and V_2. Then V_1 and V_2 are linearly independent, as is easily checked (see Exercise 14). Thus V_1 and V_2 form a basis of $\mathbb{R}^2$, so, given any point $Z_0 \in \mathbb{R}^2$, we may find a unique pair of real numbers α and β for which

$$\alpha V_1 + \beta V_2 = Z_0.$$

Now consider the function $Z(t) = \alpha X_1(t) + \beta X_2(t)$ where the $X_i(t)$ are the straight-line solutions previously. We claim that $Z(t)$ is a solution of $X' = AX$. To see this we compute

$$\begin{aligned} Z'(t) &= \alpha X_1'(t) + \beta X_2'(t) \\ &= \alpha AX_1(t) + \beta AX_2(t) \\ &= A(\alpha X_1(t) + \beta X_2(t)). \\ &= AZ(t) \end{aligned}$$

This last step follows from the linearity of matrix multiplication (see Exercise 13). Hence we have shown that $Z'(t) = AZ(t)$, so $Z(t)$ is a solution. Moreover, $Z(t)$ is a solution that satisfies $Z(0) = Z_0$. Finally, we claim that $Z(t)$ is the unique solution of $X' = AX$ that satisfies $Z(0) = Z_0$. Just as in Chapter 1, we suppose that $Y(t)$ is another such solution with $Y(0) = Z_0$. Then we may write

$$Y(t) = \zeta(t)V_1 + \mu(t)V_2$$

with $\zeta(0) = \alpha$, $\mu(0) = \beta$. Hence

$$AY(t) = Y'(t) = \zeta'(t)V_1 + \mu'(t)V_2.$$

But

$$\begin{aligned} AY(t) &= \zeta(t)AV_1 + \mu(t)AV_2 \\ &= \lambda_1\zeta(t)V_1 + \lambda_2\mu(t)V_2. \end{aligned}$$

Therefore we have

$$\begin{aligned} \zeta'(t) &= \lambda_1\zeta(t) \\ \mu'(t) &= \lambda_2\mu(t) \end{aligned}$$

with $\zeta(0) = \alpha,\ \mu(0) = \beta$. As we saw in Chapter 1, it follows that

$$\zeta(t) = \alpha e^{\lambda_1 t},\ \mu(t) = \beta e^{\lambda_2 t}$$

so that $Y(t)$ is indeed equal to $Z(t)$.

As a consequence, we have now found the unique solution to the system $X' = AX$ that satisfies $X(0) = Z_0$ for any $Z_0 \in \mathbb{R}^2$. The collection of all such solutions is called the *general solution* of $X' = AX$. That is, the general solution is the collection of solutions of $X' = AX$ that features a unique solution of the initial value problem $X(0) = Z_0$ for each $Z_0 \in \mathbb{R}^2$.

We therefore have shown the following:

Theorem. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $X' = AX$ is given by*

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2. \qquad \blacksquare$$

Example. Consider the second-order differential equation

$$x'' + 3x' + 2x = 0.$$

This is a specific case of the damped harmonic oscillator discussed earlier, where the mass is 1, the spring constant is 2, and the damping constant is 3. As a system, this equation may be rewritten:

$$X' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X = AX.$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0,$$

so the system has eigenvalues -1 and -2. The eigenvector corresponding to the eigenvalue -1 is given by solving the equation

$$(A + I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In component form this equation becomes

$$\begin{aligned} x + y &= 0 \\ -2x - 2y &= 0. \end{aligned}$$

Hence, one eigenvector associated to the eigenvalue -1 is $(1, -1)$. In similar fashion we compute that an eigenvector associated to the eigenvalue -2 is $(1, -2)$. Note that these two eigenvectors are linearly independent. Therefore, by the previous theorem, the general solution of this system is

$$X(t) = \alpha e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

That is, the position of the mass is given by the first component of the solution

$$x(t) = \alpha e^{-t} + \beta e^{-2t}$$

and the velocity is given by the second component

$$y(t) = x'(t) = -\alpha e^{-t} - 2\beta e^{-2t}. \qquad \blacksquare$$

2.7 The Linearity Principle

The theorem discussed in the previous section is a very special case of the fundamental theorem for n-dimensional linear systems, which we shall prove in Section 6.1 of Chapter 6. For the two-dimensional version of this result, note that if $X' = AX$ is a planar linear system for which $Y_1(t)$ and $Y_2(t)$ are both solutions, then just as before, the function $\alpha Y_1(t) + \beta Y_2(t)$ is also a solution of this system. We do not need real and distinct eigenvalues to prove this. This fact is known as the *linearity principle.* More importantly, if the initial conditions $Y_1(0)$ and $Y_2(0)$ are linearly independent vectors, then these vectors form a basis of $\mathbb{R}^2$. Hence, given any vector $X_0 \in \mathbb{R}^2$, we may determine constants α and β such that $X_0 = \alpha Y_1(0) + \beta Y_2(0)$. Then the linearity principle tells us that the solution $X(t)$ satisfying the initial condition $X(0) = X_0$ is given by $X(t) = \alpha Y_1(t) + \beta Y_2(t)$. Hence we have produced a solution of the system that solves any given initial value problem. The existence and uniqueness theorem for linear systems in Chapter 6 will show that this solution is also unique. This important result may then be summarized:

Theorem. *Let $X' = AX$ be a planar system. Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions of this system, and that the vectors $Y_1(0)$ and $Y_2(0)$ are linearly independent. Then*

$$X(t) = \alpha Y_1(t) + \beta Y_2(t)$$

is the unique solution of this system that satisfies $X(0) = \alpha Y_1(0) + \beta Y_2(0)$. $\blacksquare$

EXERCISES

1. Find the eigenvalues and eigenvectors of each of the following 2×2 matrices:

$$\text{(a)} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\text{(c)} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} 1 & 3 \\ \sqrt{2} & 3\sqrt{2} \end{pmatrix}$$

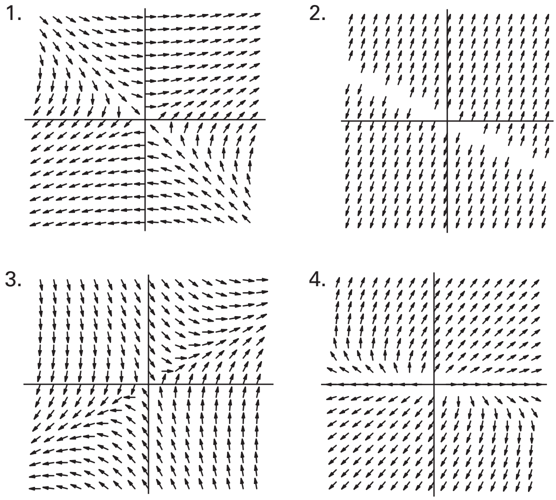


Figure 2.2 Match these direction fields with the systems in Exercise 2.

2. Find the general solution of each of the following linear systems:

$$\text{(a) } X' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} X \qquad \text{(b) } X' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} X$$

$$\text{(c) } X' = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} X \qquad \text{(d) } X' = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix} X$$

3. In Figure 2.2, you see four direction fields. Match each of these direction fields with one of the systems in the previous question.

4. Find the general solution of the system

$$X' = \begin{pmatrix} a & b \\ c & a \end{pmatrix} X$$

where $bc > 0$.

5. Find the general solution of the system

$$X' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} X.$$

6. For the harmonic oscillator system $x'' + bx' + kx = 0$, find all values of b and k for which this system has real, distinct eigenvalues. Find the general solution of this system in these cases. Find the solution of the system that satisfies the initial condition $(0, 1)$. Describe the motion of the mass in this particular case.

7. Consider the 2×2 matrix

$$A = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}.$$

Find the value a_0 of the parameter a for which A has repeated real eigenvalues. What happens to the eigenvectors of this matrix as a approaches a_0?

8. Describe all possible 2×2 matrices whose eigenvalues are 0 and 1.

9. Give an example of a linear system for which (e^{-t}, α) is a solution for every constant α. Sketch the direction field for this system. What is the general solution of this system.

10. Give an example of a system of differential equations for which $(t, 1)$ is a solution. Sketch the direction field for this system. What is the general solution of this system?

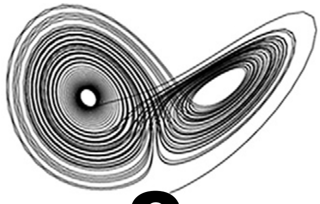
11. Prove that two vectors $V = (v_1, v_2)$ and $W = (w_1, w_2)$ are linearly independent if and only if

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0.$$

12. Prove that if λ, μ are real eigenvalues of a 2×2 matrix, then any nonzero column of the matrix $A - \lambda I$ is an eigenvector for μ.

13. Let A be a 2×2 matrix and V_1 and V_2 vectors in $\mathbb{R}^2$. Prove that $A(\alpha V_1 + \beta V_2) = \alpha A V_1 + \beta V_2$.

14. Prove that the eigenvectors of a 2×2 matrix corresponding to distinct real eigenvalues are always linearly independent.



3 Phase Portraits for Planar Systems

Given the linearity principle from the previous chapter, we may now compute the general solution of any planar system. There is a seemingly endless number of distinct cases, but we will see that these represent in the simplest possible form nearly all of the types of solutions we will encounter in the higher dimensional case.

3.1 Real Distinct Eigenvalues

Consider $X' = AX$ and suppose that A has two real eigenvalues $\lambda_1 < \lambda_2$. Assuming for the moment that $\lambda_i \neq 0$, there are three cases to consider:

1. $\lambda_1 < 0 < \lambda_2$;
2. $\lambda_1 < \lambda_2 < 0$;
3. $0 < \lambda_1 < \lambda_2$.

We give a specific example of each case; any system that falls into any one of these three categories may be handled in a similar manner, as we show later.

Example. (Saddle) First consider the simple system $X' = AX$ where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_1 < 0 < \lambda_2$. This can be solved immediately since the system decouples into two unrelated first-order equations:

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y.$$

We already know how to solve these equations, but, having in mind what comes later, let's find the eigenvalues and eigenvectors. The characteristic equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

so λ_1 and λ_2 are the eigenvalues. An eigenvector corresponding to λ_1 is $(1, 0)$ and to λ_2 is $(0, 1)$. Hence we find the general solution

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\lambda_1 < 0$, the straight-line solutions of the form $\alpha e^{\lambda_1 t}(1, 0)$ lie on the x-axis and tend to $(0, 0)$ as $t \to \infty$. This axis is called the *stable line.* Since $\lambda_2 > 0$, the solutions $\beta e^{\lambda_2 t}(0, 1)$ lie on the y-axis and tend away from $(0, 0)$ as $t \to \infty$; this axis is the *unstable line.* All other solutions (with $\alpha, \beta \neq 0$) tend to ∞ in the direction of the unstable line, as $t \to \infty$, since $X(t)$ comes closer and closer to $(0, \beta e^{\lambda_2 t})$ as t increases. In backward time, these solutions tend to ∞ in the direction of the stable line. ■

In Figure 3.1 we have plotted the *phase portrait* of this system. The phase portrait is a picture of a collection of representative solution curves of the

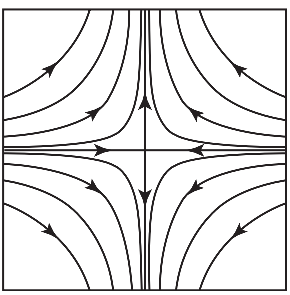


Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

system in $\mathbb{R}^2$, which we call the *phase plane*. The equilibrium point of a system of this type (eigenvalues satisfying $\lambda_1 < 0 < \lambda_2$) is called a *saddle*.

For a slightly more complicated example of this type, consider $X' = AX$ where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

As we saw in Chapter 2, the eigenvalues of A are ± 2. The eigenvector associated to $\lambda = 2$ is the vector $(3, 1)$; the eigenvector associated to $\lambda = -2$ is $(1, -1)$. Hence we have an unstable line that contains straight-line solutions of the form

$$X_1(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

each of which tends away from the origin as $t \to \infty$. The stable line contains the straight-line solutions

$$X_2(t) = \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which tend toward the origin as $t \to \infty$. By the linearity principle, any other solution assumes the form

$$X(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some α, β. Note that, if $\alpha \neq 0$, as $t \to \infty$, we have

$$X(t) \sim \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = X_1(t)$$

whereas, if $\beta \neq 0$, as $t \to -\infty$,

$$X(t) \sim \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = X_2(t).$$

Thus, as time increases, the typical solution approaches $X_1(t)$ while, as time decreases, this solution tends toward $X_2(t)$, just as in the previous case. Figure 3.2 displays this phase portrait.

In the general case where A has a positive and negative eigenvalue, we always find a similar stable and unstable line on which solutions tend toward or away

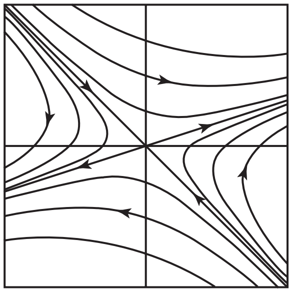


Figure 3.2 Saddle phase portrait for $x' = x + 3y$, $y' = x - y$.

from the origin. All other solutions approach the unstable line as $t \to \infty$, and tend toward the stable line as $t \to -\infty$.

Example. (Sink) Now consider the case $X' = AX$ where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

but $\lambda_1 < \lambda_2 < 0$. As above we find two straight-line solutions and then the general solution:

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Unlike the saddle case, now all solutions tend to $(0, 0)$ as $t \to \infty$. The question is: How do they approach the origin? To answer this, we compute the slope dy/dx of a solution with $\beta \neq 0$. We write

$$x(t) = \alpha e^{\lambda_1 t}$$
$$y(t) = \beta e^{\lambda_2 t}$$

and compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t}.$$

Since $\lambda_2 - \lambda_1 > 0$, it follows that these slopes approach $\pm\infty$ (provided $\beta \neq 0$). Thus these solutions tend to the origin tangentially to the y-axis. ■

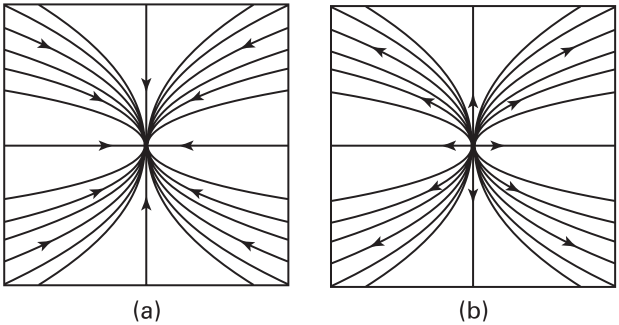


Figure 3.3 Phase portraits for (a) a sink and (b) a source.

Since $\lambda_1 < \lambda_2 < 0$, we call λ_1 the stronger eigenvalue and λ_2 the weaker eigenvalue. The reason for this in this particular case is that the x-coordinates of solutions tend to 0 much more quickly than the y-coordinates. This accounts for why solutions (except those on the line corresponding to the λ_1 eigenvector) tend to "hug" the straight-line solution corresponding to the weaker eigenvalue as they approach the origin.

The phase portrait for this system is displayed in Figure 3.3a. In this case the equilibrium point is called a *sink*.

More generally, if the system has eigenvalues $\lambda_1 < \lambda_2 < 0$ with eigenvectors (u_1, u_2) and (v_1, v_2), respectively, then the general solution is

$$\alpha e^{\lambda_1 t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The slope of this solution is given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2}{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1} \\ &= \left(\frac{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2}{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1} \right) \frac{e^{-\lambda_2 t}}{e^{-\lambda_2 t}} \\ &= \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2) t} u_2 + \lambda_2 \beta v_2}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2) t} u_1 + \lambda_2 \beta v_1}, \end{aligned}$$

which tends to the slope v_2/v_1 of the λ_2 eigenvector, unless we have $\beta = 0$. If $\beta = 0$, our solution is the straight-line solution corresponding to the eigenvalue λ_1. Hence all solutions (except those on the straight line corresponding to the stronger eigenvalue) tend to the origin tangentially to the straight-line solution corresponding to the weaker eigenvalue in this case as well.

Example. (Source) When the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

satisfies $0 < \lambda_2 < \lambda_1$, our vector field may be regarded as the negative of the previous example. The general solution and phase portrait remain the same, except that all solutions now tend away from $(0, 0)$ along the same paths. See Figure 3.3b. ■

Now one may argue that we are presenting examples here that are much too simple. While this is true, we will soon see that any system of differential equations whose matrix has real distinct eigenvalues can be manipulated into the above special forms by changing coordinates.

Finally, a special case occurs if one of the eigenvalues is equal to 0. As we have seen, there is a straight-line of equilibrium points in this case. If the other eigenvalue λ is nonzero, then the sign of λ determines whether the other solutions tend toward or away from these equilibria (see Exercises 10 and 11 at the end of this chapter).

3.2 Complex Eigenvalues

It may happen that the roots of the characteristic polynomial are complex numbers. In analogy with the real case, we call these roots *complex eigenvalues.* When the matrix A has complex eigenvalues, we no longer have straight line solutions. However, we can still derive the general solution as before by using a few tricks involving complex numbers and functions. The following examples indicate the general procedure.

Example. (Center) Consider $X' = AX$ with

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

and $\beta \neq 0$. The characteristic polynomial is $\lambda^2 + \beta^2 = 0$, so the eigenvalues are now the imaginary numbers $\pm i\beta$. Without worrying about the resulting complex vectors, we react just as before to find the eigenvector corresponding to $\lambda = i\beta$. We therefore solve

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $i\beta x = \beta y$, since the second equation is redundant. Thus we find a complex eigenvector $(1, i)$, and so the function

$$X(t) = e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

is a complex solution of $X' = AX$.

Now in general it is not polite to hand someone a complex solution to a real system of differential equations, but we can remedy this with the help of Euler's formula

$$e^{i\beta t} = \cos \beta t + i \sin \beta t.$$

Using this fact, we rewrite the solution as

$$X(t) = \begin{pmatrix} \cos \beta t + i \sin \beta t \\ i(\cos \beta t + i \sin \beta t) \end{pmatrix} = \begin{pmatrix} \cos \beta t + i \sin \beta t \\ - \sin \beta t + i \cos \beta t \end{pmatrix}.$$

Better yet, by breaking $X(t)$ into its real and imaginary parts, we have

$$X(t) = X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t)$$

where

$$X_{\mathrm{Re}}(t) = \begin{pmatrix} \cos \beta t \\ - \sin \beta t \end{pmatrix}, \; X_{\mathrm{Im}}(t) = \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}.$$

But now we see that both $X_{\mathrm{Re}}(t)$ and $X_{\mathrm{Im}}(t)$ are (real!) solutions of the original system. To see this, we simply check

$$\begin{aligned} X'_{\mathrm{Re}}(t) + iX'_{\mathrm{Im}}(t) &= X'(t) \\ &= AX(t) \\ &= A(X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t)) \\ &= AX_{\mathrm{Re}} + iAX_{\mathrm{Im}}(t). \end{aligned}$$

Equating the real and imaginary parts of this equation yields $X'_{\text{Re}} = AX_{\text{Re}}$ and $X'_{\text{Im}} = AX_{\text{Im}}$, which shows that both are indeed solutions. Moreover, since

$$X_{\text{Re}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_{\text{Im}}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the linear combination of these solutions

$$X(t) = c_1 X_{\text{Re}}(t) + c_2 X_{\text{Im}}(t)$$

where c_1 and c_2 are arbitrary constants provides a solution to any initial value problem.

We claim that this is the general solution of this equation. To prove this, we need to show that these are the only solutions of this equation. Suppose that this is not the case. Let

$$Y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

be another solution. Consider the complex function $f(t) = (u(t) + iv(t))e^{i\beta t}$. Differentiating this expression and using the fact that $Y(t)$ is a solution of the equation yields $f'(t) = 0$. Hence $u(t) + iv(t)$ is a complex constant times $e^{-i\beta t}$. From this it follows directly that $Y(t)$ is a linear combination of $X_{\text{Re}}(t)$ and $X_{\text{Im}}(t)$.

Note that each of these solutions is a periodic function with period $2\pi/\beta$. Indeed, the phase portrait shows that all solutions lie on circles centered at the origin. These circles are traversed in the clockwise direction if $\beta > 0$, counterclockwise if $\beta < 0$. See Figure 3.4. This type of system is called a *center*. ■

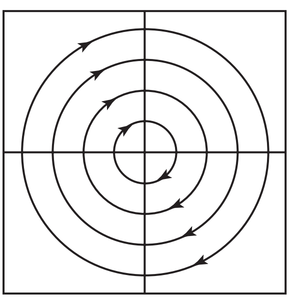


Figure 3.4 Phase portrait for a center.

Example. (Spiral Sink, Spiral Source) More generally, consider $X' = AX$ where

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and $\alpha, \beta \neq 0$. The characteristic equation is now $\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2$, so the eigenvalues are $\lambda = \alpha \pm i\beta$. An eigenvector associated to $\alpha + i\beta$ is determined by the equation

$$(\alpha - (\alpha + i\beta))x + \beta y = 0.$$

Thus $(1, i)$ is again an eigenvector. Hence we have complex solutions of the form

$$\begin{aligned} X(t) &= e^{(\alpha+i\beta)t}\begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + ie^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix} \\ &= X_{\text{Re}}(t) + iX_{\text{Im}}(t). \end{aligned}$$

As above, both $X_{\text{Re}}(t)$ and $X_{\text{Im}}(t)$ yield real solutions of the system whose initial conditions are linearly independent. Thus we find the general solution

$$X(t) = c_1 e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + c_2 e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

Without the term $e^{\alpha t}$, these solutions would wind periodically around circles centered at the origin. The $e^{\alpha t}$ term converts solutions into spirals that either spiral into the origin (when $\alpha < 0$) or away from the origin ($\alpha > 0$). In these cases the equilibrium point is called a *spiral sink* or *spiral source*, respectively. See Figure 3.5. ■

3.3 Repeated Eigenvalues

The only remaining cases occur when A has repeated real eigenvalues. One simple case occurs when A is a diagonal matrix of the form

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

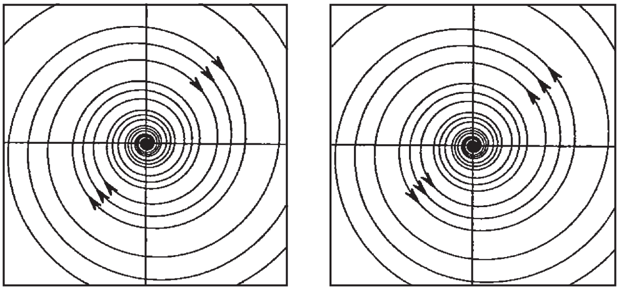


Figure 3.5 Phase portraits for a spiral sink and a spiral source.

The eigenvalues of A are both equal to λ. In this case every nonzero vector is an eigenvector since

$$AV = \lambda V$$

for any $V \in \mathbb{R}^2$. Hence solutions are of the form

$$X(t) = \alpha e^{\lambda t} V.$$

Each such solution lies on a straight line through $(0, 0)$ and either tends to $(0, 0)$ (if $\lambda < 0$) or away from $(0, 0)$ (if $\lambda > 0$). So this is an easy case.

A more interesting case occurs when

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Again both eigenvalues are equal to λ, but now there is only one linearly independent eigenvector given by $(1, 0)$. Hence we have one straight-line solution

$$X_1(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

To find other solutions, note that the system can be written

$$\begin{aligned} x' &= \lambda x + y \\ y' &= \lambda y. \end{aligned}$$

Thus, if $y \neq 0$, we must have

$$y(t) = \beta e^{\lambda t}.$$

Therefore the differential equation for $x(t)$ reads

$$x' = \lambda x + \beta e^{\lambda t}.$$

This is a nonautonomous, first-order differential equation for $x(t)$. One might first expect solutions of the form $e^{\lambda t}$, but the nonautonomous term is also in this form. As you perhaps saw in calculus, the best option is to guess a solution of the form

$$x(t) = \alpha e^{\lambda t} + \mu t e^{\lambda t}$$

for some constants α and μ. This technique is often called "the method of undetermined coefficients." Inserting this guess into the differential equation shows that $\mu = \beta$ while α is arbitrary. Hence the solution of the system may be written

$$\alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

This is in fact the general solution (see Exercise 12).

Note that, if $\lambda < 0$, each term in this solution tends to 0 as $t \to \infty$. This is clear for the $\alpha e^{\lambda t}$ and $\beta e^{\lambda t}$ terms. For the term $\beta t e^{\lambda t}$ this is an immediate consequence of l'Hôpital's rule. Hence all solutions tend to $(0, 0)$ as $t \to \infty$. When $\lambda > 0$, all solutions tend away from $(0, 0)$. See Figure 3.6. In fact, solutions tend toward or away from the origin in a direction tangent to the eigenvector $(1, 0)$ (see Exercise 7).

3.4 Changing Coordinates

Despite differences in the associated phase portraits, we really have dealt with only three types of matrices in these past three sections:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where λ may equal μ in the first case.

Any 2×2 matrix that is in one of these three forms is said to be in *canonical form.* Systems in this form may seem rather special, but they are not. Given

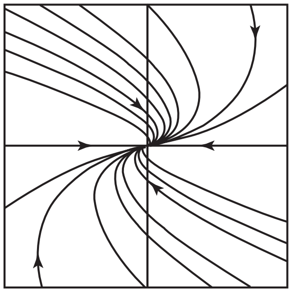


Figure 3.6 Phase portrait for a system with repeated negative eigenvalues.

any linear system $X' = AX$, we can always "change coordinates" so that the new system's coefficient matrix is in canonical form and hence easily solved. Here is how to do this.

A *linear map* (or *linear transformation*) on $\mathbb{R}^2$ is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

That is, T simply multiplies any vector by the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We will thus think of the linear map and its matrix as being interchangeable, so that we also write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We hope no confusion will result from this slight imprecision.

Now suppose that T is *invertible*. This means that the matrix T has an *inverse matrix* S that satisfies $TS = ST = I$ where I is the 2×2 identity matrix. It is traditional to denote the inverse of a matrix T by T^{-1}. As is easily checked, the matrix

$$S = \frac{1}{\det T}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

serves as T^{-1} if $\det T \neq 0$. If $\det T = 0$, then we know from Chapter 2 that there are infinitely many vectors (x, y) for which

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence there is no inverse matrix in this case, for we would need

$$\begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}T\begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for each such vector. We have shown:

Proposition. *The* 2×2 *matrix* T *is invertible if and only if* $\det T \neq 0$. ■

Now, instead of considering a linear system $X' = AX$, suppose we consider a different system

$$Y' = (T^{-1}AT)Y$$

for some invertible linear map T. Note that if $Y(t)$ is a solution of this new system, then $X(t) = TY(t)$ solves $X' = AX$. Indeed, we have

$$\begin{aligned}(TY(t))' &= TY'(t) \\ &= T(T^{-1}AT)Y(t) \\ &= A(TY(t))\end{aligned}$$

as required. That is, the linear map T converts solutions of $Y' = (T^{-1}AT)Y$ to solutions of $X' = AX$. Alternatively, T^{-1} takes solutions of $X' = AX$ to solutions of $Y' = (T^{-1}AT)Y$.

We therefore think of T as a change of coordinates that converts a given linear system into one whose coefficient matrix is different. What we hope to be able to do is find a linear map T that converts the given system into a system of the form $Y' = (T^{-1}AT)Y$ that is easily solved. And, as you may have guessed, we can always do this by finding a linear map that converts a given linear system to one in canonical form.

Example. (Real Eigenvalues) Suppose the matrix A has two real, distinct eigenvalues λ_1 and λ_2 with associated eigenvectors V_1 and V_2. Let T be the matrix whose columns are V_1 and V_2. Thus $TE_j = V_j$ for $j = 1, 2$ where the

E_j form the standard basis of $\mathbb{R}^2$. Also, $T^{-1}V_j = E_j$. Therefore we have

$$\begin{aligned}(T^{-1}AT)E_j &= T^{-1}AV_j = T^{-1}(\lambda_j V_j)\\ &= \lambda_j T^{-1}V_j\\ &= \lambda_j E_j.\end{aligned}$$

Thus the matrix $T^{-1}AT$ assumes the canonical form

$$T^{-1}AT = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

and the corresponding system is easy to solve. ■

Example. As a further specific example, suppose

$$A = \begin{pmatrix}-1 & 0\\ 1 & -2\end{pmatrix}$$

The characteristic equation is $\lambda^2 + 3\lambda + 2$, which yields eigenvalues $\lambda = -1$ and $\lambda = -2$. An eigenvector corresponding to $\lambda = -1$ is given by solving

$$(A+I)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

which yields an eigenvector $(1, 1)$. Similarly an eigenvector associated to $\lambda = -2$ is given by $(0, 1)$.

We therefore have a pair of straight-line solutions, each tending to the origin as $t \to \infty$. The straight-line solution corresponding to the weaker eigenvalue lies along the line $y = x$; the straight-line solution corresponding to the stronger eigenvalue lies on the y-axis. All other solutions tend to the origin tangentially to the line $y = x$.

To put this system in canonical form, we choose T to be the matrix whose columns are these eigenvectors:

$$T = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

so that

$$T^{-1} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

Finally, we compute

$$T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},$$

so $T^{-1}AT$ is in canonical form. The general solution of the system $Y' = (T^{-1}AT)Y$ is

$$Y(t) = \alpha e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the general solution of $X' = AX$ is

$$\begin{aligned} TY(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(\alpha e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \alpha e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus the linear map T converts the phase portrait for the system

$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} Y$$

to that of $X' = AX$ as shown in Figure 3.7. ■

Note that we really do not have to go through the step of converting a specific system to one in canonical form; once we have the eigenvalues and eigenvectors, we can simply write down the general solution. We take this extra step because, when we attempt to classify all possible linear systems, the canonical form of the system will greatly simplify this process.

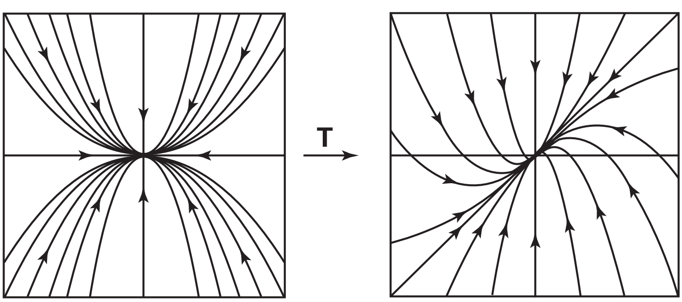


Figure 3.7 The change of variables T in the case of a (real) sink.

Example. (Complex Eigenvalues) Now suppose that the matrix A has complex eigenvalues $\alpha \pm i\beta$ with $\beta \neq 0$. Then we may find a complex eigenvector $V_1 + iV_2$ corresponding to $\alpha + i\beta$, where both V_1 and V_2 are real vectors. We claim that V_1 and V_2 are linearly independent vectors in $\mathbb{R}^2$. If this were not the case, then we would have $V_1 = cV_2$ for some $c \in \mathbb{R}$. But then we have

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2) = (\alpha + i\beta)(c + i)V_2.$$

But we also have

$$A(V_1 + iV_2) = (c + i)AV_2.$$

So we conclude that $AV_2 = (\alpha + i\beta)V_2$. This is a contradiction since the left-hand side is a real vector while the right is complex.

Since $V_1 + iV_2$ is an eigenvector associated to $\alpha + i\beta$, we have

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2).$$

Equating the real and imaginary components of this vector equation, we find

$$\begin{aligned} AV_1 &= \alpha V_1 - \beta V_2 \\ AV_2 &= \beta V_1 + \alpha V_2. \end{aligned}$$

Let T be the matrix whose columns are V_1 and V_2. Hence $TE_j = V_j$ for $j = 1, 2$. Now consider $T^{-1}AT$. We have

$$\begin{aligned} (T^{-1}AT)E_1 &= T^{-1}(\alpha V_1 - \beta V_2) \\ &= \alpha E_1 - \beta E_2 \end{aligned}$$

and similarly

$$(T^{-1}AT)E_2 = \beta E_1 + \alpha E_2.$$

Thus the matrix $T^{-1}AT$ is in the canonical form

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

We saw that the system $Y' = (T^{-1}AT)Y$ has phase portrait corresponding to a spiral sink, center, or spiral source depending on whether $\alpha < 0$, $\alpha = 0$, or

$\alpha > 0$. Therefore the phase portrait of $X' = AX$ is equivalent to one of these after changing coordinates using T. ■

Example. (Another Harmonic Oscillator) Consider the second-order equation

$$x'' + 4x = 0.$$

This corresponds to an undamped harmonic oscillator with mass 1 and spring constant 4. As a system, we have

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X = AX.$$

The characteristic equation is

$$\lambda^2 + 4 = 0$$

so that the eigenvalues are $\pm 2i$. A complex eigenvector associated to $\lambda = 2i$ is a solution of the system

$$\begin{aligned} -2ix + y &= 0 \\ -4x - 2iy &= 0. \end{aligned}$$

One such solution is the vector $(1, 2i)$. So we have a complex solution of the form

$$e^{2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Breaking this solution into its real and imaginary parts, we find the general solution

$$X(t) = c_1 \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

Thus the position of this oscillator is given by

$$x(t) = c_1 \cos 2t + c_2 \sin 2t,$$

which is a periodic function of period π.

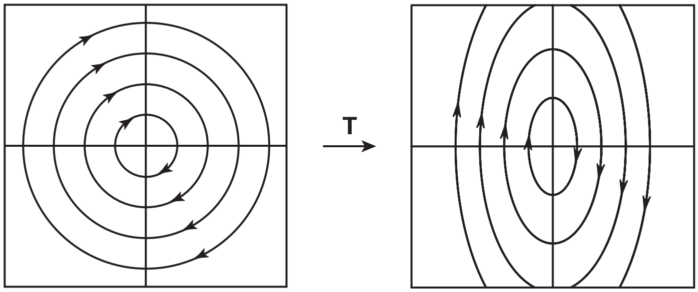


Figure 3.8 The change of variables T in the case of a center.

Now, let T be the matrix whose columns are the real and imaginary parts of the eigenvector $(1, 2i)$. That is

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, we compute easily that

$$T^{-1}AT = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

which is in canonical form. The phase portraits of these systems are shown in Figure 3.8. Note that T maps the circular solutions of the system $Y' = (T^{-1}AT)Y$ to elliptic solutions of $X' = AX$. ■

Example. (Repeated Eigenvalues) Suppose A has a single real eigenvalue λ. If there exist a pair of linearly independent eigenvectors, then in fact A must be in the form

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

so the system $X' = AX$ is easily solved (see Exercise 15).

For the more complicated case, let's assume that V is an eigenvector and that every other eigenvector is a multiple of V. Let W be any vector for which V and W are linearly independent. Then we have

$$AW = \mu V + \nu W$$

for some constants $\mu, \nu \in \mathbb{R}$. Note that $\mu \neq 0$, for otherwise we would have a second linearly independent eigenvector W with eigenvalue ν. We claim that $\nu = \lambda$. If $\nu - \lambda \neq 0$, a computation shows that

$$A\left(W + \left(\frac{\mu}{\nu - \lambda}\right) V\right) = \nu\left(W + \left(\frac{\mu}{\nu - \lambda}\right) V\right).$$

This says that ν is a second eigenvalue different from λ. Hence we must have $\nu = \lambda$.

Finally, let $U = (1/\mu)W$. Then

$$AU = V + \frac{\lambda}{\mu} W = V + \lambda U.$$

Thus if we define $TE_1 = V,\ TE_2 = U$, we get

$$T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

as required. Thus $X' = AX$ is again in canonical form after this change of coordinates. ■

EXERCISES

1. In Figure 3.9 on page 58, you see six phase portraits. Match each of these phase portraits with one of the following linear systems:

(a) $\begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & -2 \\ 5 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -2 \\ 5 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$ (f) $\begin{pmatrix} -3 & 5 \\ -2 & 2 \end{pmatrix}$

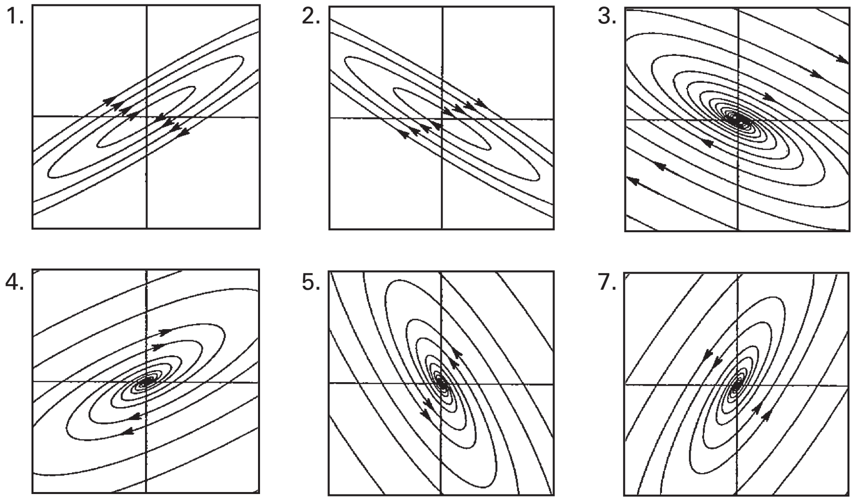


Figure 3.9 Match these phase portraits with the systems in Exercise 1.

2. For each of the following systems of the form $X' = AX$

(a) Find the eigenvalues and eigenvectors of A.
(b) Find the matrix T that puts A in canonical form.
(c) Find the general solution of both $X' = AX$ and $Y' = (T^{-1}AT)Y$.
(d) Sketch the phase portraits of both systems.

$$\text{(i)}\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(ii)}\ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{(iii)}\ A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{(iv)}\ A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\text{(v)}\ A = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \qquad \text{(vi)}\ A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

3. Find the general solution of the following harmonic oscillator equations:

(a) $x'' + x' + x = 0$
(b) $x'' + 2x' + x = 0$

4. Consider the harmonic oscillator system

$$X' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} X$$

where $b \geq 0$, $k > 0$ and the mass $m = 1$.

(a) For which values of k, b does this system have complex eigenvalues? Repeated eigenvalues? Real and distinct eigenvalues?

(b) Find the general solution of this system in each case.

(c) Describe the motion of the mass when the mass is released from the initial position $x = 1$ with zero velocity in each of the cases in part (a).

5. Sketch the phase portrait of $X' = AX$ where

$$A = \begin{pmatrix} a & 1 \\ 2a & 2 \end{pmatrix}.$$

For which values of a do you find a bifurcation? Describe the phase portrait for a-values above and below the bifurcation point.

6. Consider the system

$$X' = \begin{pmatrix} 2a & b \\ b & 0 \end{pmatrix} X.$$

Sketch the regions in the ab–plane where this system has different types of canonical forms.

7. Consider the system

$$X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X$$

with $\lambda \neq 0$. Show that all solutions tend to (respectively, away from) the origin tangentially to the eigenvector $(1, 0)$ when $\lambda < 0$ (respectively, $\lambda > 0$).

8. Find all 2×2 matrices that have pure imaginary eigenvalues. That is, determine conditions on the entries of a matrix that guarantee that the matrix has pure imaginary eigenvalues.

9. Determine a computable condition that guarantees that, if a matrix A has complex eigenvalues, then solutions of $X' = AX$ travel around the origin in the counterclockwise direction.

10. Consider the system

$$X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X$$

where $a + d \neq 0$ but $ad - bc = 0$. Find the general solution of this system and sketch the phase portrait.

11. Find the general solution and describe completely the phase portrait for

$$X' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X.$$

12. Prove that

$$\alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

is the general solution of

$$X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X.$$

13. Prove that a 2×2 matrix A always satisfies its own characteristic equation. That is, if $\lambda^2 + \alpha\lambda + \beta = 0$ is the characteristic equation associated to A, then the matrix $A^2 + \alpha A + \beta I$ is the 0 matrix.

14. Suppose the 2×2 matrix A has repeated eigenvalues λ. Let $V \in \mathbb{R}^2$. Using the previous problem, show that either V is an eigenvector for A or else $(A - \lambda I)V$ is an eigenvector for A.

15. Suppose the matrix A has repeated real eigenvalues λ and there exists a pair of linearly independent eigenvectors associated to A. Prove that

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

16. Consider the (nonlinear) system

$$x' = |y|$$
$$y' = -x.$$

Use the methods of this chapter to describe the phase portrait.

4
Classification of Planar Systems

In this chapter, we summarize what we have accomplished so far using a dynamical systems point of view. Among other things, this means that we would like to have a complete "dictionary" of all possible behaviors of 2×2 autonomous linear systems. One of the dictionaries we present here is geometric: the trace-determinant plane. The other dictionary is more dynamic: this involves the notion of conjugate systems.

4.1 The Trace-Determinant Plane

For a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we know that the eigenvalues are the roots of the characteristic equation, which can be written

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

The constant term in this equation is $\det A$. The coefficient of λ also has a name: The quantity $a + d$ is called the *trace* of A and is denoted by $\operatorname{tr} A$.

Thus the eigenvalues satisfy

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

and are given by

$$\lambda_{\pm} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}\right).$$

Note that $\lambda_+ + \lambda_- = \operatorname{tr} A$ and $\lambda_+\lambda_- = \det A$, so the trace is the sum of the eigenvalues of A while the determinant is the product of the eigenvalues of A. We will also write $T = \operatorname{tr} A$ and $D = \det A$. Knowing T and D tells us the eigenvalues of A and therefore virtually everything about the geometry of solutions of $X' = AX$. For example, the values of T and D tell us whether solutions spiral into or away from the origin, whether we have a center, and so forth.

We may display this classification visually by painting a picture in the *trace-determinant plane*. In this picture a matrix with trace T and determinant D corresponds to the point with coordinates (T, D). The location of this point in the TD–plane then determines the geometry of the phase portrait as above. For example, the sign of $T^2 - 4D$ tells us that the eigenvalues are:

1. Complex with nonzero imaginary part if $T^2 - 4D < 0$;
2. Real and distinct if $T^2 - 4D > 0$;
3. Real and repeated if $T^2 - 4D = 0$.

Thus the location of (T, D) relative to the parabola $T^2 - 4D = 0$ in the TD–plane tells us all we need to know about the eigenvalues of A from an algebraic point of view.

In terms of phase portraits, however, we can say more. If $T^2 - 4D < 0$, then the real part of the eigenvalues is $T/2$, and so we have a

1. Spiral sink if $T < 0$;
2. Spiral source if $T > 0$;
3. Center if $T = 0$.

If $T^2 - 4D > 0$ we have a similar breakdown into cases. In this region, both eigenvalues are real. If $D < 0$, then we have a saddle. This follows since D is the product of the eigenvalues, one of which must be positive, the other negative. Equivalently, if $D < 0$, we compute

$$T^2 < T^2 - 4D$$

so that

$$\pm T < \sqrt{T^2 - 4D}.$$

Thus we have

$$T + \sqrt{T^2 - 4D} > 0$$
$$T - \sqrt{T^2 - 4D} < 0$$

so the eigenvalues are real and have different signs. If $D > 0$ and $T < 0$ then both

$$T \pm \sqrt{T^2 - 4D} < 0,$$

so we have a (real) sink. Similarly, $T > 0$ and $D > 0$ leads to a (real) source.

When $D = 0$ and $T \neq 0$, we have one zero eigenvalue, while both eigenvalues vanish if $D = T = 0$.

Plotting all of this verbal information in the *TD*–plane gives us a visual summary of all of the different types of linear systems. The equations above partition the *TD*–plane into various regions in which systems of a particular type reside. See Figure 4.1. This yields a geometric classification of 2×2 linear systems.

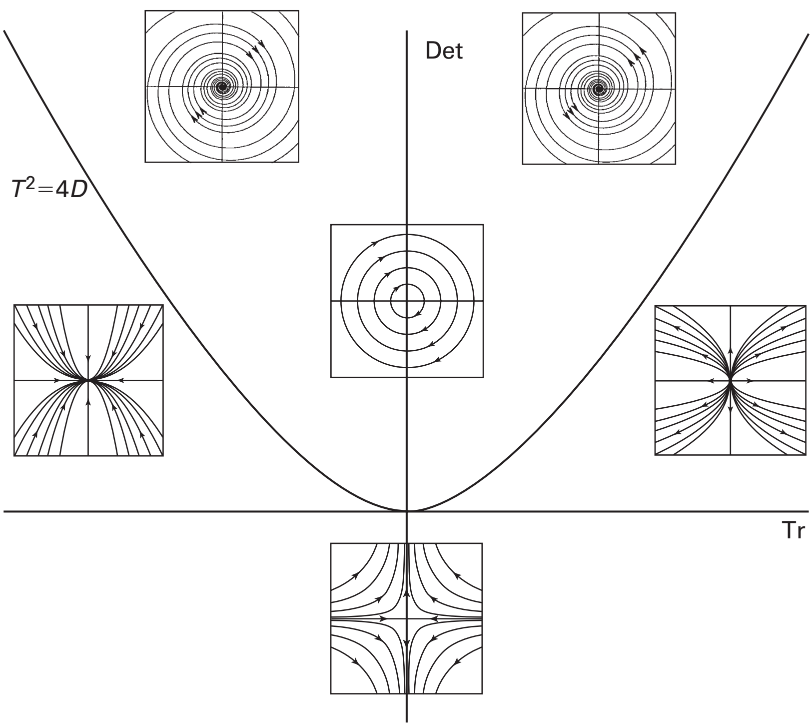


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

A couple of remarks are in order. First, the trace-determinant plane is a two-dimensional representation of what is really a four-dimensional space, since 2×2 matrices are determined by four parameters, the entries of the matrix. Thus there are infinitely many different matrices corresponding to each point in the TD–plane. While all of these matrices share the same eigenvalue configuration, there may be subtle differences in the phase portraits, such as the direction of rotation for centers and spiral sinks and sources, or the possibility of one or two independent eigenvectors in the repeated eigenvalue case.

We also think of the trace-determinant plane as the analog of the bifurcation diagram for planar linear systems. A one-parameter family of linear systems corresponds to a curve in the TD–plane. When this curve crosses the T-axis, the positive D-axis, or the parabola $T^2 - 4D = 0$, the phase portrait of the linear system undergoes a bifurcation: A major change occurs in the geometry of the phase portrait.

Finally, note that we may obtain quite a bit of information about the system from D and T without ever computing the eigenvalues. For example, if $D < 0$, we know that we have a saddle at the origin. Similarly, if both D and T are positive, then we have a source at the origin.

4.2 Dynamical Classification

In this section we give a different, more dynamical classification of planar linear systems. From a dynamical systems point of view, we are usually interested primarily in the long-term behavior of solutions of differential equations. Thus two systems are equivalent if their solutions share the same fate. To make this precise we recall some terminology introduced in Section 1.5 of Chapter 1.

To emphasize the dependence of solutions on both time and the initial conditions X_0, we let $\phi_t(X_0)$ denote the solution that satisfies the initial condition X_0. That is, $\phi_0(X_0) = X_0$. The function $\phi(t, X_0) = \phi_t(X_0)$ is called the *flow* of the differential equation, whereas ϕ_t is called the *time t map* of the flow.

For example, let

$$X' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} X.$$

Then the time t map is given by

$$\phi_t(x_0, y_0) = \left(x_0 e^{2t}, y_0 e^{3t}\right).$$

Thus the flow is a function that depends on both time and the initial values.

We will consider two systems to be dynamically equivalent if there is a function h that takes one flow to the other. We require that this function be a *homeomorphism*, that is, h is a one-to-one, onto, and continuous function whose inverse is also continuous.

Definition
Suppose $X' = AX$ and $X' = BX$ have flows ϕ^A and ϕ^B. These two systems are (topologically) *conjugate* if there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)).$$

The homeomorphism h is called a *conjugacy*. Thus a conjugacy takes the solution curves of $X' = AX$ to those of $X' = BX$.

Example. For the one-dimensional linear differential equations

$$x' = \lambda_1 x \quad \text{and} \quad x' = \lambda_2 x$$

we have the flows

$$\phi^j(t, x_0) = x_0 e^{\lambda_j t}$$

for $j = 1, 2$. Suppose that λ_1 and λ_2 are nonzero and have the same sign. Then let

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1} & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} & \text{if } x < 0 \end{cases}$$

where we recall that

$$x^{\lambda_2/\lambda_1} = \exp\left(\frac{\lambda_2}{\lambda_1}\log(x)\right).$$

Note that h is a homeomorphism of the real line. We claim that h is a conjugacy between $x' = \lambda_1 x$ and $x' = \lambda_2 x$. To see this, we check that when $x_0 > 0$

$$h(\phi^1(t, x_0)) = \left(x_0 e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1}$$

$$= x_0^{\lambda_2/\lambda_1} e^{\lambda_2 t}$$
$$= \phi^2(t, h(x_0))$$

as required. A similar computation works when $x_0 < 0$. ■

There are several things to note here. First, λ_1 and λ_2 must have the same sign, because otherwise we would have $|h(0)| = \infty$, in which case h is not a homeomorphism. This agrees with our notion of dynamical equivalence: If λ_1 and λ_2 have the same sign, then their solutions behave similarly as either both tend to the origin or both tend away from the origin. Also, note that if $\lambda_2 < \lambda_1$, then h is not differentiable at the origin, whereas if $\lambda_2 > \lambda_1$ then $h^{-1}(x) = x^{\lambda_1/\lambda_2}$ is not differentiable at the origin. This is the reason why we require h to be only a homeomorphism and not a *diffeomorphism* (a differentiable homeomorphism with differentiable inverse): If we assume differentiability, then we must have $\lambda_1 = \lambda_2$, which does not yield a very interesting notion of "equivalence."

This gives a classification of (autonomous) linear first-order differential equations, which agrees with our qualitative observations in Chapter 1. There are three conjugacy "classes": the sinks, the sources, and the special "in-between" case, $x' = 0$, where all solutions are constants.

Now we move to the planar version of this scenario. We first note that we only need to decide on conjugacies among systems whose matrices are in canonical form. For, as we saw in Chapter 3, if the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ puts A in canonical form, then T takes the time t map of the flow of $Y' = (T^{-1}AT)Y$ to the time t map for $X' = AX$.

Our classification of planar linear systems now proceeds just as in the one-dimensional case. We will stay away from the case where the system has eigenvalues with real part equal to 0, but you will tackle this case in the exercises.

Definition

A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is *hyperbolic.*

Theorem. *Suppose that the 2×2 matrices A_1 and A_2 are hyperbolic. Then the linear systems $X' = A_iX$ are conjugate if and only if each matrix has the same number of eigenvalues with negative real part.* ■

Thus two hyperbolic matrices yield conjugate linear systems if both sets of eigenvalues fall into the same category below:

1. One eigenvalue is positive and the other is negative;

2. Both eigenvalues have negative real parts;
3. Both eigenvalues have positive real parts.

Before proving this, note that this theorem implies that a system with a spiral sink is conjugate to a system with a (real) sink. Of course! Even though their phase portraits look very different, it is nevertheless the case that all solutions of both systems share the same fate: They tend to the origin as $t \to \infty$.

Proof: Recall from the previous discussion that we may assume all systems are in canonical form. Then the proof divides into three distinct cases.

Case 1

Suppose we have two linear systems $X' = A_i X$ for $i = 1, 2$ such that each A_i has eigenvalues $\lambda_i < 0 < \mu_i$. Thus each system has a saddle at the origin. This is the easy case. As we saw previously, the real differential equations $x' = \lambda_i x$ have conjugate flows via the homeomorphism

$$h_1(x) = \begin{cases} x^{\lambda_2/\lambda_1} & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} & \text{if } x < 0 \end{cases}.$$

Similarly, the equations $y' = \mu_i y$ also have conjugate flows via an analogous function h_2. Now define

$$H(x, y) = (h_1(x), h_2(y)).$$

Then one checks immediately that H provides a conjugacy between these two systems.

Case 2

Consider the system $X' = AX$ where A is in canonical form with eigenvalues that have negative real parts. We further assume that the matrix A is not in the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda < 0$. Thus, in canonical form, A assumes one of the following two forms:

$$\text{(a)} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $\alpha, \lambda, \mu < 0$. We will show that, in either case, the system is conjugate to $X' = BX$ where

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It then follows that any two systems of this form are conjugate.

Consider the unit circle in the plane parameterized by the curve $X(\theta) = (\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$. We denote this circle by S^1. We first claim that the vector field determined by a matrix in the above form must point inside S^1. In case (a), we have that the vector field on S^1 is given by

$$AX(\theta) = \begin{pmatrix} \alpha\cos\theta + \beta\sin\theta \\ -\beta\cos\theta + \alpha\sin\theta \end{pmatrix}.$$

The outward pointing normal vector to S^1 at $X(\theta)$ is

$$N(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

The dot product of these two vectors satisfies

$$AX(\theta) \cdot N(\theta) = \alpha(\cos^2\theta + \sin^2\theta) < 0$$

since $\alpha < 0$. This shows that $AX(\theta)$ does indeed point inside S^1. Case (b) is even easier.

As a consequence, each nonzero solution of $X' = AX$ crosses S^1 exactly once. Let ϕ_t^A denote the time t map for this system, and let $\tau = \tau(x, y)$ denote the time at which $\phi_t^A(x, y)$ meets S^1. Thus

$$\left|\phi^A_{\tau(x,y)}(x, y)\right| = 1.$$

Let ϕ_t^B denote the time t map for the system $X' = BX$. Clearly,

$$\phi_t^B(x, y) = (e^{-t}x, e^{-t}y).$$

We now define a conjugacy H between these two systems. If $(x, y) \neq (0, 0)$, let

$$H(x, y) = \phi^B_{-\tau(x,y)}\phi^A_{\tau(x,y)}(x, y)$$

and set $H(0, 0) = (0, 0)$. Geometrically, the value of $H(x, y)$ is given by following the solution curve of $X' = AX$ exactly $\tau(x, y)$ time units (forward

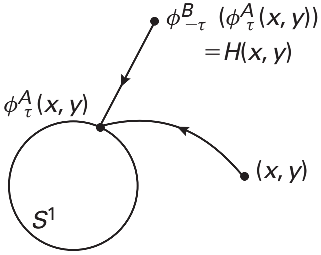


Figure 4.2 The definition of $\tau\,(x, y)$.

or backward) until the solution reaches S^1, and then following the solution of $X' = BX$ starting at that point on S^1 and proceeding in the opposite time direction exactly τ time units. See Figure 4.2.

To see that H gives a conjugacy, note first that

$$\tau\left(\phi^A_s(x,y)\right) = \tau(x,y) - s$$

since

$$\phi^A_{\tau-s}\phi^A_s(x,y) = \phi^A_\tau(x,y) \in S^1.$$

Therefore we have

$$\begin{aligned} H\left(\phi^A_s(x,y)\right) &= \phi^B_{-\tau+s}\phi^A_{\tau-s}\left(\phi^A_s(x,y)\right) \\ &= \phi^B_s\phi^B_{-\tau}\phi^A_\tau(x,y) \\ &= \phi^B_s\left(H(x,y)\right). \end{aligned}$$

So H is a conjugacy.

Now we show that H is a homeomorphism. We can construct an inverse for H by simply reversing the process defining H. That is, let

$$G(x,y) = \phi^A_{-\tau_1(x,y)}\phi^B_{\tau_1(x,y)}(x,y)$$

and set $G(0,0) = (0,0)$. Here $\tau_1(x,y)$ is the time for the solution of $X' = BX$ through (x,y) to reach S^1. An easy computation shows that $\tau_1(x,y) = \log r$ where $r^2 = x^2 + y^2$. Clearly, $G = H^{-1}$ so H is one to one and onto. Also, G is continuous at $(x,y) \neq (0,0)$ since G may be written

$$G(x,y) = \phi^A_{-\log r}\left(\frac{x}{r}, \frac{y}{r}\right),$$

which is a composition of continuous functions. For continuity of G at the origin, suppose that (x, y) is close to the origin, so that r is small. Observe that as $r \to 0$, $-\log r \to \infty$. Now $(x/r, y/r)$ is a point on S^1 and for r sufficiently small, $\phi^A_{-\log r}$ maps the unit circle very close to $(0, 0)$. This shows that G is continuous at $(0, 0)$.

We thus need only show continuity of H. For this, we need to show that $\tau(x, y)$ is continuous. But τ is determined by the equation

$$\left|\phi^A_t(x, y)\right| = 1.$$

We write $\phi^A_t(x, y) = (x(t), y(t))$. Taking the partial derivative of $|\phi^A_t(x, y)|$ with respect to t, we find

$$\begin{aligned}
\frac{\partial}{\partial t}\left|\phi^A_t(x, y)\right| &= \frac{\partial}{\partial t}\sqrt{(x(t))^2 + (y(t))^2} \\
&= \frac{1}{\sqrt{(x(t))^2 + (y(t))^2}}\left(x(t)x'(t) + y(t)y'(t)\right) \\
&= \frac{1}{\left|\phi^A_t(x, y)\right|}\left(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}\right).
\end{aligned}$$

But the latter dot product is nonzero when $t = \tau(x, y)$ since the vector field given by $(x'(t), y'(t))$ points inside S^1. Hence

$$\frac{\partial}{\partial t}\left|\phi^A_t(x, y)\right| \neq 0$$

at $(\tau(x, y), x, y)$. Thus we may apply the implicit function theorem to show that τ is differentiable at (x, y) and hence continuous. Continuity of H at the origin follows as in the case of $G = H^{-1}$. Thus H is a homeomorphism and we have a conjugacy between $X' = AX$ and $X' = BX$.

Note that this proof works equally well if the eigenvalues have positive real parts.

Case 3

Finally, suppose that

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda < 0$. The associated vector field need not point inside the unit circle in this case. However, if we let

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix},$$

then the vector field given by

$$Y' = (T^{-1}AT)Y$$

now does have this property, provided $\epsilon > 0$ is sufficiently small. Indeed

$$T^{-1}AT = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$$

so that

$$\left(T^{-1}AT\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right) \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \lambda + \epsilon \sin\theta \cos\theta.$$

Thus if we choose $\epsilon < -\lambda$, this dot product is negative. Therefore the change of variables T puts us into the situation where the same proof as in Case 2 applies. This completes the proof. ■

4.3 Exploration: A 3D Parameter Space

Consider the three-parameter family of linear systems given by

$$X' = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} X$$

where a, b, and c are parameters.

1. First, fix $a > 0$. Describe the analog of the trace-determinant plane in the bc–plane. That is, identify the bc–values in this plane where the corresponding system has saddles, centers, spiral sinks, etc. Sketch these regions in the bc–plane.
2. Repeat the previous question when $a < 0$ and when $a = 0$.
3. Describe the bifurcations that occur as a changes from positive to negative.
4. Now put all of the previous information together and give a description of the full three-dimensional parameter space for this system. You could build a 3D model of this space, create a flip-book animation of the changes

as, say, a varies, or use a computer model to visualize this image. In any event, your model should accurately capture all of the distinct regions in this space.

EXERCISES

1. Consider the one-parameter family of linear systems given by

$$X' = \begin{pmatrix} a & \sqrt{2} + (a/2) \\ \sqrt{2} - (a/2) & 0 \end{pmatrix} X.$$

(a) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies.

(b) Discuss any bifurcations that occur along this path and compute the corresponding values of a.

2. Sketch the analog of the trace-determinant plane for the two-parameter family of systems

$$X' = \begin{pmatrix} a & b \\ b & a \end{pmatrix} X$$

in the ab–plane. That is, identify the regions in the ab–plane where this system has similar phase portraits.

3. Consider the harmonic oscillator equation (with $m = 1$)

$$x'' + bx' + kx = 0$$

where $b \geq 0$ and $k > 0$. Identify the regions in the relevant portion of the bk–plane where the corresponding system has similar phase portraits.

4. Prove that $H(x, y) = (x, -y)$ provides a conjugacy between

$$X' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X \quad \text{and} \quad Y' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} Y.$$

5. For each of the following systems, find an explicit conjugacy between their flows.

(a) $X' = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} X$ and $Y' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} Y$

(b) $X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X$ and $Y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} Y.$

6. Prove that any two linear systems with the same eigenvalues $\pm i\beta$, $\beta \neq 0$ are conjugate. What happens if the systems have eigenvalues $\pm i\beta$ and $\pm i\gamma$ with $\beta \neq \gamma$? What if $\gamma = -\beta$?
7. Consider all linear systems with exactly one eigenvalue equal to 0. Which of these systems are conjugate? Prove this.
8. Consider all linear systems with two zero eigenvalues. Which of these systems are conjugate? Prove this.
9. Provide a complete description of the conjugacy classes for 2×2 systems in the nonhyperbolic case.

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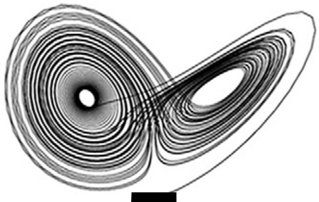


5 Higher Dimensional Linear Algebra

As in Chapter 2, we need to make another detour into the world of linear algebra before proceeding to the solution of higher dimensional linear systems of differential equations. There are many different canonical forms for matrices in higher dimensions, but most of the algebraic ideas involved in changing coordinates to put matrices into these forms are already present in the 2×2 case. In particular, the case of matrices with distinct (real or complex) eigenvalues can be handled with minimal additional algebraic complications, so we deal with this case first. This is the "generic case," as we show in Section 5.6. Matrices with repeated eigenvalues demand more sophisticated concepts from linear algebra; we provide this background in Section 5.4. We assume throughout this chapter that the reader is familiar with solving systems of linear algebraic equations by putting the associated matrix in (reduced) row echelon form.

5.1 Preliminaries from Linear Algebra

In this section we generalize many of the algebraic notions of Section 2.3 to higher dimensions. We denote a vector $X \in \mathbb{R}^n$ in coordinate form as

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

In the plane, we called a pair of vectors V and W linearly independent if they were not collinear. Equivalently, V and W were linearly independent if there were no (nonzero) real numbers α and β such that $\alpha V + \beta W$ was the zero vector.

More generally, in $\mathbb{R}^n$, a collection of vectors $V_1, \ldots, V_k$ in $\mathbb{R}^n$ is said to be *linearly independent* if, whenever

$$\alpha_1 V_1 + \cdots + \alpha_k V_k = 0$$

with $\alpha_j \in \mathbb{R}$, it follows that each $\alpha_j = 0$. If we can find such $\alpha_1, \ldots, \alpha_k$, not all of which are 0, then the vectors are *linearly dependent*. Note that if $V_1, \ldots, V_k$ are linearly independent and W is the linear combination

$$W = \beta_1 V_1 + \cdots + \beta_k V_k,$$

then the β_j are unique. This follows since, if we could also write

$$W = \gamma_1 V_1 + \cdots + \gamma_k V_k,$$

then we would have

$$0 = W - W = (\beta_1 - \gamma_1)V_1 + \cdots (\beta_k - \gamma_k)V_k,$$

which forces $\beta_j = \gamma_j$ for each j, by linear independence of the V_j.

Example. The vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are clearly linearly independent in $\mathbb{R}^3$. More generally, let E_j be the vector in $\mathbb{R}^n$ whose jth component is 1 and all other components are 0. Then the vectors $E_1, \ldots, E_n$ are linearly independent in $\mathbb{R}^n$. The collection of vectors $E_1, \ldots, E_n$ is called the *standard basis* of $\mathbb{R}^n$. We will discuss the concept of a basis in Section 5.4. ■

Example. The vectors $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ in $\mathbb{R}^3$ are also linearly independent, because if we have

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then the third component says that $\alpha_3 = 0$. The fact that $\alpha_3 = 0$ in the second component then says that $\alpha_2 = 0$, and finally the first component similarly tells us that $\alpha_1 = 0$. On the other hand, the vectors $(1, 1, 1)$, $(1, 2, 3)$, and $(2, 3, 4)$ are linearly dependent, for we have

$$1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

■

When solving linear systems of differential equations, we will often encounter special subsets of $\mathbb{R}^n$ called *subspaces.* A *subspace* of $\mathbb{R}^n$ is a collection of all possible linear combinations of a given set of vectors. More precisely, given $V_1, \ldots, V_k \in \mathbb{R}^n$, the set

$$\mathcal{S} = \{\alpha_1 V_1 + \cdots + \alpha_k V_k \mid \alpha_j \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^n$. In this case we say that $\mathcal{S}$ is *spanned* by $V_1, \ldots, V_k$. Equivalently, it can be shown (see Exercise 12 at the end of this chapter) that a subspace $\mathcal{S}$ is a nonempty subset of $\mathbb{R}^n$ having the following two properties:

1. If $X, Y \in \mathcal{S}$, then $X + Y \in \mathcal{S}$;
2. If $X \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $\alpha X \in \mathcal{S}$.

Note that the zero vector lies in every subspace of $\mathbb{R}^n$ and that any linear combination of vectors in a subspace $\mathcal{S}$ also lies in $\mathcal{S}$.

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. Indeed, any vector V in $\mathcal{P}$ may be written in the form $(x, y, -x - y)$ or

$$V = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

which shows that the vectors $(1, 0, -1)$ and $(0, 1, -1)$ span $\mathcal{P}$. ■

In linear algebra, one often encounters rectangular $n \times m$ matrices, but in differential equations, most often these matrices are square ($n \times n$). Consequently we will assume that all matrices in this chapter are $n \times n$. We write such a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

more compactly as $A = [a_{ij}]$.

For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define the product AX to be the vector

$$AX = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix},$$

so that the ith entry in this vector is the dot product of the ith row of A with the vector X.

Matrix sums are defined in the obvious way. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ matrices, then we define $A + B = C$ where $C = [a_{ij} + b_{ij}]$. Matrix arithmetic has some obvious linearity properties:

1. $A(k_1X_1 + k_2X_2) = k_1AX_1 + k_2AX_2$ where $k_j \in \mathbb{R}$, $X_j \in \mathbb{R}^n$;
2. $A + B = B + A$;
3. $(A + B) + C = A + (B + C)$.

The product of the $n \times n$ matrices A and B is defined to be the $n \times n$ matrix $AB = [c_{ij}]$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

so that c_{ij} is the dot product of the ith row of A with the jth column of B. We can easily check that, if A, B, and C are $n \times n$ matrices, then

1. $(AB)C = A(BC)$;
2. $A(B + C) = AB + AC$;
3. $(A + B)C = AC + BC$;
4. $k(AB) = (kA)B = A(kB)$ for any $k \in \mathbb{R}$.

All of the above properties of matrix arithmetic are easily checked by writing out the ij entries of the corresponding matrices. It is important to remember that matrix multiplication is not commutative, so that $AB \neq BA$ in general. For example

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

whereas

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Also, matrix cancellation is usually forbidden; if $AB = AC$, then we do not necessarily have $B = C$ as in

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

In particular, if AB is the zero matrix, it does not follow that one of A or B is also the zero matrix.

The $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix C for which $AC = CA = I$ where I is the $n \times n$ identity matrix that has 1s along the diagonal and 0s elsewhere. The matrix C is called the *inverse* of A. Note that if A has an inverse, then this inverse is unique. For if $AB = BA = I$ as well, then

$$C = CI = C(AB) = (CA)B = IB = B.$$

The inverse of A is denoted by A^{-1}.

If A is invertible, then the vector equation $AX = V$ has a unique solution for any $V \in \mathbb{R}^n$. Indeed, $A^{-1}V$ is one solution. Moreover, it is the only one, for if Y is another solution, then we have

$$Y = (A^{-1}A)Y = A^{-1}(AY) = A^{-1}V.$$

For the converse of this statement, recall that the equation $AX = V$ has unique solutions if and only if the *reduced row echelon form* of the matrix A is the identity matrix. The reduced row echelon form of A is obtained by applying to A a sequence of *elementary row operations* of the form

1. Add k times row i of A to row j;
2. Interchange row i and j;
3. Multiply row i by $k \neq 0$.

Note that these elementary row operations correspond exactly to the operations that are used to solve linear systems of algebraic equations:

1. Add k times equation i to equation j;
2. Interchange equations i and j;
3. Multiply equation i by $k \neq 0$.

Each of these elementary row operations may be represented by multiplying A by an *elementary* matrix. For example, if $L = [\ell_{ij}]$ is the matrix that has 1's along the diagonal, $\ell_{ji} = k$ for some choice of i and j, $i \neq j$, and all other entries 0, then LA is the matrix that is obtained by performing row operation 1 on A. Similarly, if L has 1's along the diagonal with the exception that $\ell_{ii} = \ell_{jj} = 0$, but $\ell_{ij} = \ell_{ji} = 1$, and all other entries are 0, then LA is the matrix that results after performing row operation 2 on A. Finally, if L is the identity matrix with a k instead of 1 in the ii position, then LA is the matrix obtained by performing row operation 3. A matrix L in one of these three forms is called an elementary matrix.

Each elementary matrix is invertible, since its inverse is given by the matrix that simply "undoes" the corresponding row operation. As a consequence, any product of elementary matrices is invertible. Therefore, if $L_1, \ldots, L_n$ are the elementary matrices that correspond to the row operations that put

A into the reduced row echelon form, which is the identity matrix, then $(L_n \cdot \cdots \cdot L_1) = A^{-1}$. That is, if the vector equation $AX = V$ has unique solutions for any $V \in \mathbb{R}^n$, then A is invertible. Thus we have our first important result.

Proposition. *Let A be an $n \times n$ matrix. Then the system of algebraic equations $AX = V$ has a unique solution for any $V \in \mathbb{R}^n$ if and only if A is invertible.* ■

Thus the natural question now is: How do we tell if A is invertible? One answer is provided by the following result.

Proposition. *The matrix A is invertible if and only if the columns of A form a linearly independent set of vectors.*

Proof: Suppose first that A is invertible and has columns $V_1, \ldots, V_n$. We have $AE_j = V_j$ where the E_j form the standard basis of $\mathbb{R}^n$. If the V_j are not linearly independent, we may find real numbers $\alpha_1, \ldots, \alpha_n$, not all zero, such that $\sum_j \alpha_j V_j = 0$. But then

$$0 = \sum_{j=1}^{n} \alpha_j AE_j = A\left(\sum_{j=1}^{n} \alpha_j E_j\right).$$

Hence the equation $AX = 0$ has two solutions, the nonzero vector $(\alpha_1, \ldots, \alpha_n)$ and the 0 vector. This contradicts the previous proposition.

Conversely, suppose that the V_j are linearly independent. If A is not invertible, then we may find a pair of vectors X_1 and X_2 with $X_1 \neq X_2$ and $AX_1 = AX_2$. Therefore the nonzero vector $Z = X_1 - X_2$ satisfies $AZ = 0$. Let $Z = (\alpha_1, \ldots, \alpha_n)$. Then we have

$$0 = AZ = \sum_{j=1}^{n} \alpha_j V_j,$$

so that the V_j are not linearly independent. This contradiction establishes the result. ■

A more computable criterion for determining whether or not a matrix is invertible, as in the 2×2 case, is given by the determinant of A. Given the $n \times n$ matrix A, we will denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

Definition
The *determinant* of $A = [a_{ij}]$ is defined inductively by

$$\det A = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det A_{1k}.$$

Note that we know the determinant of a 2×2 matrix, so this induction makes sense for $k > 2$.

Example. From the definition we compute

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$
$$= -3 + 12 - 9 = 0.$$

We remark that the definition of $\det A$ given above involves "expanding along the first row" of A. One can equally well expand along the jth row so that

$$\det A = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det A_{jk}.$$

We will not prove this fact; the proof is an entirely straightforward though tedious calculation. Similarly, $\det A$ can be calculated by expanding along a given column (see Exercise 1). ■

Example. Expanding the matrix in the previous example along the second and third rows yields the same result:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = -4 \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 5 \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} - 6 \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$
$$= 24 - 60 + 36 = 0$$
$$= 7 \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} - 8 \det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} + 9 \det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$
$$= -21 + 48 - 27 = 0.$$

Incidentally, note that this matrix is not invertible, since

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

■

The determinant of certain types of matrices is easy to compute. A matrix $[a_{ij}]$ is called upper triangular if all entries below the main diagonal are 0. That is, $a_{ij} = 0$ if $i > j$. Lower triangular matrices are defined similarly. We have

Proposition. *If A is an upper or lower triangular $n \times n$ matrix, then* $\det A$ *is the product of the entries along the diagonal. That is,* $\det[a_{ij}] = a_{11} \dots a_{nn}$. ■

The proof is a straightforward application of induction. The following proposition describes the effects that elementary row operations have on the determinant of a matrix.

Proposition. *Let A and B be $n \times n$ matrices.*

1. *Suppose the matrix B is obtained by adding a multiple of one row of A to another row of A. Then* $\det B = \det A$.
2. *Suppose B is obtained by interchanging two rows of A. Then* $\det B = -\det A$.
3. *Suppose B is obtained by multiplying each element of a row of A by k. Then* $\det B = k \det A$.

Proof: The proof of the proposition is straightforward when A is a 2×2 matrix, so we use induction. Suppose A is $k \times k$ with $k > 2$. To compute $\det B$, we expand along a row that is left untouched by the row operation. By induction on k, we see that $\det B$ is a sum of determinants of size $(k-1) \times (k-1)$. Each of these subdeterminants has precisely the same row operation performed on it as in the case of the full matrix. By induction, it follows that each of these subdeterminants is multiplied by $1, -1$, or k in cases 1, 2, and 3, respectively. Hence $\det B$ has the same property. ■

In particular, we note that if L is an elementary matrix, then

$$\det(LA) = (\det L)(\det A).$$

Indeed, $\det L = 1, -1$, or k in cases 1–3 above (see Exercise 7). The preceding proposition now yields a criterion for A to be invertible:

Corollary. (Invertibility Criterion) *The matrix A is invertible if and only if* $\det A \neq 0$.

Proof: By elementary row operations, we can manipulate any matrix A into an upper triangular matrix. Then A is invertible if and only if all diagonal entries of this row reduced matrix are nonzero. In particular, the determinant of this matrix is nonzero. Now, by the previous observation, row operations multiply $\det A$ by nonzero numbers, so we see that all of the diagonal entries are nonzero if and only if $\det A$ is also nonzero. This concludes the proof. ■

We conclude this section with a further important property of determinants.

Proposition. $\det(AB) = (\det A)(\det B)$.

Proof: If either A or B is noninvertible, then AB is also noninvertible (see Exercise 11). Hence the proposition is true since both sides of the equation are zero. If A is invertible, then we can write

$$A = L_1 \dots L_n \cdot I$$

where each L_j is an elementary matrix. Hence

$$\begin{aligned}
\det(AB) &= \det(L_1 \dots L_n B) \\
&= \det(L_1) \det(L_2 \dots L_n B) \\
&= \det(L_1)(\det L_2) \dots (\det L_n)(\det B) \\
&= \det(L_1 \dots L_n) \det(B) \\
&= \det(A) \det(B).
\end{aligned}$$

■

5.2 Eigenvalues and Eigenvectors

As we saw in Chapter 3, eigenvalues and eigenvectors play a central role in the process of solving linear systems of differential equations.

Definition

A vector V is an *eigenvector* of an $n \times n$ matrix A if V is a nonzero solution to the system of linear equations $(A - \lambda I)V = 0$. The quantity λ is called an *eigenvalue* of A, and V is an eigenvector associated to λ.

Just as in Chapter 2, the eigenvalues of a matrix A may be real or complex and the associated eigenvectors may have complex entries.

By the invertibility criterion of the previous section, it follows that λ is an eigenvalue of A if and only if λ is a root of the *characteristic equation*

$$\det(A - \lambda I) = 0.$$

Since A is $n \times n$, this is a polynomial equation of degree n, which therefore has exactly n roots (counted with multiplicity).

As we saw in $\mathbb{R}^2$, there are many different types of solutions of systems of differential equations, and these types depend on the configuration of the eigenvalues of A and the resulting canonical forms. There are many, many more types of canonical forms in higher dimensions. We will describe these types in this and the following sections, but we will relegate some of the more specialized proofs of these facts to the exercises.

Suppose first that $\lambda_1, \ldots, \lambda_\ell$ are real and distinct eigenvalues of A with associated eigenvectors $V_1, \ldots, V_\ell$. Here "distinct" means that no two of the eigenvalues are equal. Thus $AV_k = \lambda_k V_k$ for each k. We claim that the V_k are linearly independent. If not, we may choose a maximal subset of the V_i that are linearly independent, say, $V_1, \ldots, V_j$. Then any other eigenvector may be written in a unique way as a linear combination of $V_1, \ldots, V_j$. Say V_{j+1} is one such eigenvector. Then we may find α_i, not all 0, such that

$$V_{j+1} = \alpha_1 V_1 + \cdots + \alpha_j V_j.$$

Multiplying both sides of this equation by A, we find

$$\begin{aligned} \lambda_{j+1} V_{j+1} &= \alpha_1 A V_1 + \cdots + \alpha_j A V_j \\ &= \alpha_1 \lambda_1 V_1 + \cdots + \alpha_j \lambda_j V_j. \end{aligned}$$

Now $\lambda_{j+1} \neq 0$ for otherwise we would have

$$\alpha_1 \lambda_1 V_1 + \cdots + \alpha_j \lambda_j V_j = 0,$$

with each $\lambda_i \neq 0$. This contradicts the fact that $V_1, \ldots, V_j$ are linearly independent. Hence we have

$$V_{j+1} = \alpha_1 \frac{\lambda_1}{\lambda_{j+1}} V_1 + \cdots + \alpha_j \frac{\lambda_j}{\lambda_{j+1}} V_j.$$

Since the λ_i are distinct, we have now written V_{j+1} in two different ways as a linear combination of $V_1, \ldots, V_j$. This contradicts the fact that this set of vectors is linearly independent. We have proved:

Proposition. *Suppose $\lambda_1, \ldots, \lambda_\ell$ are real and distinct eigenvalues for A with associated eigenvectors $V_1, \ldots, V_\ell$. Then the V_j are linearly independent.* ■

Of primary importance when we return to differential equations is the

Corollary. *Suppose A is an $n \times n$ matrix with real, distinct eigenvalues. Then there is a matrix T such that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

where all of the entries off the diagonal are 0.

Proof: Let V_j be an eigenvector associated to λ_j. Consider the linear map T for which $TE_j = V_j$, where the E_j form the standard basis of $\mathbb{R}^n$. That is, T is the matrix whose columns are $V_1, \ldots, V_n$. Since the V_j are linearly independent, T is invertible and we have

$$\begin{aligned}(T^{-1}AT)E_j &= T^{-1}AV_j \\ &= \lambda_j T^{-1}V_j \\ &= \lambda_j E_j.\end{aligned}$$

That is, the jth column of $T^{-1}AT$ is just the vector $\lambda_j E_j$, as required. ■

Example. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Expanding $\det(A - \lambda I)$ along the first column, we find that the characteristic equation of A is

$$\begin{aligned}\det(A - \lambda I) &= (1 - \lambda)\det\begin{pmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{pmatrix} \\ &= (1 - \lambda)((3 - \lambda)(-2 - \lambda) + 4) \\ &= (1 - \lambda)(\lambda - 2)(\lambda + 1),\end{aligned}$$

so the eigenvalues are 2, 1, and -1. The eigenvector corresponding to $\lambda = 2$ is given by solving the equations $(A - 2I)X = 0$, which yields

$$\begin{aligned}-x + 2y - z &= 0 \\ y - 2z &= 0 \\ 2y - 4z &= 0.\end{aligned}$$

These equations reduce to

$$x - 3z = 0$$
$$y - 2z = 0.$$

Hence $V_1 = (3, 2, 1)$ is an eigenvector associated to $\lambda = 2$. In similar fashion we find that $(1, 0, 0)$ is an eigenvector associated to $\lambda = 1$, while $(0, 1, 2)$ is an eigenvector associated to $\lambda = -1$. Then we set

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

A simple calculation shows that

$$AT = T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since det $T = -3$, T is invertible and we have

$$T^{-1}AT = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

■

5.3 Complex Eigenvalues

Now we treat the case where A has nonreal (complex) eigenvalues. Suppose $\alpha + i\beta$ is an eigenvalue of A with $\beta \neq 0$. Since the characteristic equation for A has real coefficients, it follows that if $\alpha + i\beta$ is an eigenvalue, then so is its complex conjugate $\overline{\alpha + i\beta} = \alpha - i\beta$.

Another way to see this is the following. Let V be an eigenvector associated to $\alpha + i\beta$. Then the equation

$$AV = (\alpha + i\beta)V$$

shows that V is a vector with complex entries. We write

$$V = \begin{pmatrix} x_1 + iy_n \\ \vdots \\ x_n + iy_n \end{pmatrix}.$$

Let $\overline{V}$ denote the complex conjugate of V:

$$\overline{V} = \begin{pmatrix} x_1 - iy_1 \\ \vdots \\ x_n - iy_n \end{pmatrix}.$$

Then we have

$$A\overline{V} = \overline{AV} = (\overline{\alpha + i\beta})\overline{V} = (\alpha - i\beta)\overline{V},$$

which shows that $\overline{V}$ is an eigenvector associated to the eigenvalue $\alpha - i\beta$.

Notice that we have (temporarily) stepped out of the "real" world of $\mathbb{R}^n$ and into the world $\mathbb{C}^n$ of complex vectors. This is not really a problem, since all of the previous linear algebraic results hold equally well for complex vectors.

Now suppose that A is a $2n \times 2n$ matrix with distinct nonreal eigenvalues $\alpha_j \pm i\beta_j$ for $j = 1, \ldots, n$. Let V_j and $\overline{V_j}$ denote the associated eigenvectors. Then, just as in the previous proposition, this collection of eigenvectors is linearly independent. That is, if we have

$$\sum_{j=1}^{n} (c_j V_j + d_j \overline{V_j}) = 0$$

where the c_j and d_j are now complex numbers, then we must have $c_j = d_j = 0$ for each j.

Now we change coordinates to put A into canonical form. Let

$$W_{2j-1} = \frac{1}{2}\big(V_j + \overline{V_j}\big)$$
$$W_{2j} = \frac{-i}{2}\big(V_j - \overline{V_j}\big).$$

Note that W_{2j-1} and W_{2j} are both real vectors. Indeed, W_{2j-1} is just the real part of V_j while W_{2j} is its imaginary part. So working with the W_j brings us back home to $\mathbb{R}^n$.

Proposition. *The vectors $W_1, \ldots, W_{2n}$ are linearly independent.*

Proof: Suppose not. Then we can find real numbers c_j, d_j for $j = 1, \ldots, n$ such that

$$\sum_{j=1}^{n} \big(c_j W_{2j-1} + d_j W_{2j}\big) = 0$$

but not all of the c_j and d_j are zero. But then we have

$$\frac{1}{2}\sum_{j=1}^{n}\left(c_j(V_j+\overline{V_j})-id_j(V_j-\overline{V_j})\right)=0$$

from which we find

$$\sum_{j=1}^{n}\left((c_j-id_j)V_j+(c_j+id_j)\overline{V_j}\right)=0.$$

Since the V_j and $\overline{V_j}$'s are linearly independent, we must have $c_j \pm id_j = 0$, from which we conclude $c_j = d_j = 0$ for all j. This contradiction establishes the result. ∎

Note that we have

$$\begin{aligned} AW_{2j-1} &= \frac{1}{2}(AV_j + A\overline{V_j}) \\ &= \frac{1}{2}\left((\alpha+i\beta)V_j + (\alpha-i\beta)\overline{V_j}\right) \\ &= \frac{\alpha}{2}(V_j+\overline{V_j}) + \frac{i\beta}{2}(V_j-\overline{V_j}) \\ &= \alpha W_{2j-1} - \beta W_{2j}. \end{aligned}$$

Similarly, we compute

$$AW_{2j} = \beta W_{2j-1} + \alpha W_{2j}.$$

Now consider the linear map T for which $TE_j = W_j$ for $j = 1,\ldots,2n$. That is, the matrix associated to T has columns $W_1,\ldots,W_{2n}$. Note that this matrix has real entries. Since the W_j are linearly independent, it follows from Section 5.2 that T is invertible. Now consider the matrix $T^{-1}AT$. We have

$$\begin{aligned} (T^{-1}AT)E_{2j-1} &= T^{-1}AW_{2j-1} \\ &= T^{-1}(\alpha W_{2j-1} - \beta W_{2j}) \\ &= \alpha E_{2j-1} - \beta E_{2j} \end{aligned}$$

and similarly

$$(T^{-1}AT)E_{2j} = \beta E_{2j-1} + \alpha W_{2j}.$$

Therefore the matrix associated to $T^{-1}AT$ is

$$T^{-1}AT = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix}$$

where each D_j is a 2×2 matrix of the form

$$D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}.$$

This is our canonical form for matrices with distinct nonreal eigenvalues.

Combining the results of this and the previous section, we have:

Theorem. *Suppose that the $n \times n$ matrix A has distinct eigenvalues. Then there exists a linear map T so that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & D_1 & & \\ & & & & \ddots & \\ & & & & & D_\ell \end{pmatrix}$$

where the D_j are 2×2 matrices in the form

$$D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}.$$

■

5.4 Bases and Subspaces

To deal with the case of a matrix with repeated eigenvalues, we need some further algebraic concepts. Recall that the collection of all linear combinations of a given finite set of vectors is called a *subspace* of $\mathbb{R}^n$. More precisely, given $V_1, \ldots, V_k \in \mathbb{R}^n$, the set

$$\mathcal{S} = \{\alpha_1 V_1 + \cdots + \alpha_k V_k \mid \alpha_j \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^n$. In this case we say that $\mathcal{S}$ is *spanned* by $V_1, \ldots, V_k$.

Definition
Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$. A collection of vectors $V_1, \ldots, V_k$ is a *basis* of $\mathcal{S}$ if the V_j are linearly independent and span $\mathcal{S}$.

Note that a subspace always has a basis, for if $\mathcal{S}$ is spanned by $V_1, \ldots, V_k$, we can always throw away certain of the V_j in order to reach a linearly independent subset of these vectors that spans $\mathcal{S}$. More precisely, if the V_j are not linearly independent, then we may find one of these vectors, say, V_k, for which

$$V_k = \beta_1 V_1 + \cdots + \beta_{k-1} V_{k-1}.$$

Hence we can write any vector in $\mathcal{S}$ as a linear combination of the $V_1, \ldots, V_{k-1}$ alone; the vector V_k is extraneous. Continuing in this fashion, we eventually reach a linearly independent subset of the V_j that spans $\mathcal{S}$.

More important for our purposes is:

Proposition. *Every basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ has the same number of elements.*

Proof: We first observe that the system of k linear equations in $k+\ell$ unknowns given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1\,k+\ell}x_{k+\ell} &= 0 \\ &\vdots \\ a_{k1}x_1 + \cdots + a_{k\,k+\ell}x_{k+\ell} &= 0 \end{aligned}$$

always has a nonzero solution if $\ell > 0$. Indeed, using row reduction, we may first solve for one unknown in terms of the others, and then we may eliminate this unknown to obtain a system of $k-1$ equations in $k+\ell-1$ unknowns. Thus we are finished by induction (the first case, $k = 1$, being obvious).

Now suppose that $V_1, \ldots, V_k$ is a basis for the subspace $\mathcal{S}$. Suppose that $W_1, \ldots, W_{k+\ell}$ is also a basis of $\mathcal{S}$, with $\ell > 0$. Then each W_j is a linear combination of the V_i, so we have constants a_{ij} such that

$$W_j = \sum_{i=1}^{k} a_{ij} V_i, \quad \text{for } j = 1, \ldots, k+\ell.$$

By the previous observation, the system of k equations

$$\sum_{j=1}^{k+\ell} a_{ij}x_j = 0, \quad \text{for } i = 1, \ldots, k$$

has a nonzero solution $(c_1, \ldots, c_{k+\ell})$. Then

$$\sum_{j=1}^{k+\ell} c_j W_j = \sum_{j=1}^{k+\ell} c_j \left(\sum_{i=1}^{k} a_{ij} V_i \right) = \sum_{i=1}^{k} \left(\sum_{j=1}^{k+\ell} a_{ij} c_j \right) V_i = 0$$

so that the W_j are linearly dependent. This contradiction completes the proof. ■

As a consequence of this result, we may define the *dimension* of a subspace $\mathcal{S}$ as the number of vectors that form any basis for $\mathcal{S}$. In particular, $\mathbb{R}^n$ is a subspace of itself, and its dimension is clearly n. Furthermore, any other subspace of $\mathbb{R}^n$ must have dimension less than n, for otherwise we would have a collection of more than n vectors in $\mathbb{R}^n$ that are linearly independent. This cannot happen by the previous proposition. The set consisting of only the 0 vector is also a subspace, and we define its dimension to be zero. We write $\dim \mathcal{S}$ for the dimension of the subspace $\mathcal{S}$.

Example. A straight line through the origin in $\mathbb{R}^n$ forms a one-dimensional subspace of $\mathbb{R}^n$, since any vector on this line may be written uniquely as tV where $V \in \mathbb{R}^n$ is a fixed nonzero vector lying on the line and $t \in \mathbb{R}$ is arbitrary. Clearly, the single vector V forms a basis for this subspace. ■

Example. The plane $\mathcal{P}$ in $\mathbb{R}^3$ defined by

$$x + y + z = 0$$

is a two-dimensional subspace of $\mathbb{R}^3$. The vectors $(1, 0, -1)$ and $(0, 1, -1)$ both lie in $\mathcal{P}$ and are linearly independent. If $W \in \mathcal{P}$, we may write

$$W = \begin{pmatrix} x \\ y \\ -y - x \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

so these vectors also span $\mathcal{P}$. ■

As in the planar case, we say that a function $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is linear if $T(X) = AX$ for some $n \times n$ matrix A. T is called a *linear map* or *linear transformation.* Using the properties of matrices discussed in Section 5.1, we have

$$T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$$

for any $\alpha, \beta \in \mathbb{R}$ and $X, Y \in \mathbb{R}^n$. We say that the linear map T is invertible if the matrix A associated to T has an inverse.

For the study of linear systems of differential equations, the most important types of subspaces are the kernels and ranges of linear maps. We define the *kernel* of T, denoted Ker T, to be the set of vectors mapped to 0 by T. The *range* of T consists of all vectors W for which there exists a vector V for which $TV = W$. This, of course, is a familiar concept from calculus. The difference here is that the range of T is always a subspace of $\mathbb{R}^n$.

Example. Consider the linear map

$$T(X) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} X.$$

If $X = (x, y, z)$, then

$$T(X) = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}.$$

Hence Ker T consists of all vectors of the form $(\alpha, 0, 0)$ while Range T is the set of vectors of the form $(\beta, \gamma, 0)$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Both sets are clearly subspaces. ■

Example. Let

$$T(X) = AX = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} X.$$

For Ker T, we seek vectors X that satisfy $AX = 0$. Using row reduction, we find that the reduced row echelon form of A is the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the solutions $X = (x, y, z)$ of $AX = 0$ satisfy $x = z$, $y = -2z$. Therefore any vector in Ker T is of the form $(z, -2z, z)$, so Ker T has dimension one. For Range T, note that the columns of A are vectors in Range T, since they are the images of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. These vectors are not linearly independent since

$$-1\begin{pmatrix}1\\4\\7\end{pmatrix} + 2\begin{pmatrix}2\\5\\8\end{pmatrix} = \begin{pmatrix}3\\6\\9\end{pmatrix}.$$

However, $(1, 4, 7)$ and $(2, 5, 8)$ are linearly independent, so these two vectors give a basis of Range T. ■

Proposition. *Let* $T\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a linear map. Then* Ker T *and* Range T *are both subspaces of* $\mathbb{R}^n$*. Moreover,*

$$\dim \operatorname{Ker} T + \dim \operatorname{Range} T = n.$$

Proof: First suppose that Ker $T = \{0\}$. Let $E_1, \ldots, E_n$ be the standard basis of $\mathbb{R}^n$. Then we claim that $TE_1, \ldots, TE_n$ are linearly independent. If this is not the case, then we may find $\alpha_1, \ldots, \alpha_n$, not all 0, such that

$$\sum_{j=1}^{n} \alpha_j TE_j = 0.$$

But then we have

$$T\left(\sum_{j=1}^{n} \alpha_j E_j\right) = 0,$$

which implies that $\sum \alpha_j E_j \in \operatorname{Ker} T$, so that $\sum \alpha_j E_j = 0$. Hence each $\alpha_j = 0$, which is a contradiction. Thus the vectors TE_j are linearly independent. But then, given $V \in \mathbb{R}^n$, we may write

$$V = \sum_{j=1}^{n} \beta_j TE_j$$

for some $\beta_1, \ldots, \beta_n$. Hence

$$V = T\left(\sum_{j=1}^{n} \beta_j E_j\right)$$

which shows that Range $T = \mathbb{R}^n$. Hence both Ker T and Range T are subspaces of $\mathbb{R}^n$ and we have dim Ker $T = 0$ and dim Range $T = n$.

If Ker $T \neq \{0\}$, we may find a nonzero vector $V_1 \in$ Ker T. Clearly, $T(\alpha V_1) = 0$ for any $\alpha \in \mathbb{R}$, so all vectors of the form αV_1 lie in Ker T. If Ker T contains additional vectors, choose one and call it V_2. Then Ker T contains all linear combinations of V_1 and V_2, since

$$T(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 TV_1 + \alpha_2 TV_2 = 0.$$

Continuing in this fashion we obtain a set of linearly independent vectors that span Ker T, thus showing that Ker T is a subspace. Note that this process must end, since every collection of more than n vectors in $\mathbb{R}^n$ is linearly dependent. A similar argument works to show that Range T is a subspace.

Now suppose that $V_1, \ldots, V_k$ form a basis of Ker T where $0 < k < n$ (the case where $k = n$ being obvious). Choose vectors $W_{k+1}, \ldots, W_n$ so that $V_1, \ldots, V_k, W_{k+1}, \ldots, W_n$ form a basis of $\mathbb{R}^n$. Let $Z_j = TW_j$ for each j. Then the vectors Z_j are linearly independent, for if we had

$$\alpha_{k+1} Z_{k+1} + \cdots + \alpha_n Z_n = 0,$$

then we would also have

$$T(\alpha_{k+1} W_{k+1} + \cdots + \alpha_n W_n) = 0.$$

This implies that

$$\alpha_{k+1} W_{k+1} + \cdots + \alpha_n W_n \in \text{Ker } T.$$

But this is impossible, since we cannot write any W_j (and hence any linear combination of the W_j) as a linear combination of the V_i. This proves that the sum of the dimensions of Ker T and Range T is n. ■

We remark that it is easy to find a set of vectors that spans Range T; simply take the set of vectors that comprise the columns of the matrix associated to T. This works since the ith column vector of this matrix is the image of the standard basis vector E_i under T. In particular, if these column vectors are linearly independent, then Ker $T = \{0\}$ and there is a unique solution to the equation $T(X) = V$ for every $V \in \mathbb{R}^n$. Hence we have:

Corollary. *If* $T: \mathbb{R}^n \to \mathbb{R}^n$ *is a linear map with* $\dim \operatorname{Ker} T = 0$, *then* T *is invertible.* ■

5.5 Repeated Eigenvalues

In this section we describe the canonical forms that arise when a matrix has repeated eigenvalues. Rather than spending an inordinate amount of time developing the general theory in this case, we will give the details only for 3×3 and 4×4 matrices with repeated eigenvalues. More general cases are relegated to the exercises. We justify this omission in the next section where we show that the "typical" matrix has distinct eigenvalues and hence can be handled as in the previous section. (If you happen to meet a random matrix while walking down the street, the chances are very good that this matrix will have distinct eigenvalues!) The most general result regarding matrices with repeated eigenvalues is given by:

Proposition. *Let* A *be an* $n \times n$ *matrix. Then there is a change of coordinates* T *for which*

$$T^{-1}AT = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}$$

where each of the B_j*'s is a square matrix (and all other entries are zero) of one of the following forms:*

$$\text{(i)} \quad \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \qquad \text{(ii)} \quad \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix}$$

where

$$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and where $\alpha, \beta, \lambda \in \mathbb{R}$ *with* $\beta \neq 0$. *The special cases where* $B_j = (\lambda)$ *or*

$$B_j = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

are, of course, allowed. ■

We first consider the case of $\mathbb{R}^3$. If A has repeated eigenvalues in $\mathbb{R}^3$, then all eigenvalues must be real. There are then two cases. Either there are two distinct eigenvalues, one of which is repeated, or else all eigenvalues are the same. The former case can be handled by a process similar to that described in Chapter 3, so we restrict our attention here to the case where A has a single eigenvalue λ of multiplicity 3.

Proposition. *Suppose* A *is a* 3×3 *matrix for which* λ *is the only eigenvalue. Then we may find a change of coordinates* T *such that* $T^{-1}AT$ *assumes one of the following three forms:*

$$\text{(i)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Proof: Let K be the kernel of $A - \lambda I$. Any vector in K is an eigenvector of A. There are then three subcases depending on whether the dimension of K is 1, 2, or 3.

If the dimension of K is 3, then $(A - \lambda I)V = 0$ for any $V \in \mathbb{R}^3$. Hence $A = \lambda I$. This yields matrix (i).

Suppose the dimension of K is 2. Let R be the range of $A - \lambda I$. Then R has dimension 1 since $\dim K + \dim R = 3$, as we saw in the previous section. We claim that $R \subset K$. If this is not the case, let $V \in R$ be a nonzero vector. Since $(A - \lambda I)V \in R$ and R is one dimensional, we must have $(A - \lambda I)V = \mu V$ for some $\mu \neq 0$. But then $AV = (\lambda + \mu)V$, so we have found a new eigenvalue $\lambda + \mu$. This contradicts our assumption, so we must have $R \subset K$.

Now let $V_1 \in R$ be nonzero. Since $V_1 \in K$, V_1 is an eigenvector and so $(A - \lambda I)V_1 = 0$. Since V_1 also lies in R, we may find $V_2 \in \mathbb{R}^3 - K$ with $(A - \lambda I)V_2 = V_1$. Since K is two dimensional we may choose a second vector $V_3 \in K$ such that V_1 and V_3 are linearly independent. Note that V_3 is also an eigenvector. If we now choose the change of coordinates $TE_j = V_j$ for $j = 1, 2, 3$, then it follows easily that $T^{-1}AT$ assumes the form of case (ii).

Finally, suppose that K has dimension 1. Thus R has dimension 2. We claim that, in this case, $K \subset R$. If this is not the case, then $(A - \lambda I)R = R$ and so

$A - \lambda I$ is invertible on R. Thus, if $V \in R$, there is a unique $W \in R$ for which $(A - \lambda I)W = V$. In particular, we have

$$\begin{aligned} AV &= A(A - \lambda I)W \\ &= (A^2 - \lambda A)W \\ &= (A - \lambda I)(AW). \end{aligned}$$

This shows that, if $V \in R$, then so too is AV. Hence A also preserves the subspace R. It then follows immediately that A must have an eigenvector in R, but this then says that $K \subset R$ and we have a contradiction.

Next we claim that $(A - \lambda I)R = K$. To see this, note that $(A - \lambda I)R$ is one dimensional, since $K \subset R$. If $(A - \lambda I)R \neq K$, there is a nonzero vector $V \notin K$ for which $(A - \lambda I)R = \{tV\}$ where $t \in \mathbb{R}$. But then $(A - \lambda I)V = tV$ for some $t \in \mathbb{R}$, $t \neq 0$, and so $AV = (t + \lambda)V$ yields another new eigenvalue. Thus we must in fact have $(A - \lambda I)R = K$.

Now let $V_1 \in K$ be an eigenvector for A. As above there exists $V_2 \in R$ such that $(A - \lambda I)V_2 = V_1$. Since $V_2 \in R$ there exists V_3 such that $(A - \lambda I)V_3 = V_2$. Note that $(A - \lambda I)^2 V_3 = V_1$. The V_j are easily seen to be linearly independent. Moreover, the linear map defined by $TE_j = V_j$ finally puts A into canonical form (iii). This completes the proof. ■

Example. Suppose

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Expanding along the first row, we find

$$\det(A - \lambda I) = (2 - \lambda)[(2 - \lambda)^2 + 1] - (2 - \lambda) = (2 - \lambda)^3$$

so the only eigenvalue is 2. Solving $(A - 2I)V = 0$ yields only one independent eigenvector $V_1 = (1, -1, 0)$, so we are in case (iii) of the proposition. We compute

$$(A - 2I)^2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that the vector $V_3 = (1, 0, 0)$ solves $(A - 2I)^2 V_3 = V_1$. We also have

$$(A - 2I)V_3 = V_2 = (0, 0, -1).$$

As before, we let $TE_j = V_j$ for $j = 1, 2, 3$, so that

$$T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then $T^{-1}AT$ assumes the canonical form

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

■

Example. Now suppose

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}.$$

Again expanding along the first row, we find

$$\det(A - \lambda I) = (1 - \lambda)[(3 - \lambda)(2 - \lambda)] + (2 - \lambda) = (2 - \lambda)^3$$

so again the only eigenvalue is 2. This time, however, we have

$$A - 2I = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

so that we have two linearly independent eigenvectors (x, y, z) for which we must have $x = y$ while z is arbitrary. Note that $(A - 2I)^2$ is the zero matrix, so we may choose any vector that is not an eigenvector as V_2, say, $V_2 = (1, 0, 0)$. Then $(A - 2I)V_2 = V_1 = (-1, -1, -1)$ is an eigenvector. A second linearly independent eigenvector is then $V_3 = (0, 0, 1)$, for example. Defining $TE_j = V_j$ as usual then yields the canonical form

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

■

Now we turn to the 4×4 case. The case of all real eigenvalues is similar to the 3×3 case (though a little more complicated algebraically) and is left as an exercise. Thus we assume that A has repeated complex eigenvalues $\alpha \pm i\beta$ with $\beta \neq 0$.

There are just two cases; either we can find a pair of linearly independent eigenvectors corresponding to $\alpha + i\beta$, or we can find only one such eigenvector. In the former case, let V_1 and V_2 be the independent eigenvectors. The $\overline{V_1}$ and $\overline{V_2}$ are linearly independent eigenvectors for $\alpha - i\beta$. As before, choose the real vectors

$$\begin{aligned}
W_1 &= (V_1 + \overline{V_1})/2 \\
W_2 &= -i(V_1 - \overline{V_1})/2 \\
W_3 &= (V_2 + \overline{V_2})/2 \\
W_4 &= -i(V_2 - \overline{V_2})/2.
\end{aligned}$$

If we set $TE_j = W_j$ then changing coordinates via T puts A in canonical form

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}.$$

If we find only one eigenvector V_1 for $\alpha + i\beta$, then we solve the system of equations $(A - (\alpha + i\beta)I)X = V_1$ as in the case of repeated real eigenvalues. The proof of the previous proposition shows that we can always find a nonzero solution V_2 of these equations. Then choose the W_j as above and set $TE_j = W_j$. Then T puts A into the canonical form

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}.$$

For example, we compute

$$\begin{aligned}
(T^{-1}AT)E_3 &= T^{-1}AW_3 \\
&= T^{-1}A(V_2 + \overline{V_2})/2 \\
&= T^{-1}\big((V_1 + (\alpha + i\beta)V_2)/2 + (\overline{V_1} + (\alpha - i\beta)\overline{V_2})/2\big) \\
&= T^{-1}\big((V_1 + \overline{V_1})/2 + \alpha\big(V_2 + \overline{V_2}\big)/2 + i\beta\big(V_2 - \overline{V_2}\big)/2\big) \\
&= E_1 + \alpha E_3 - \beta E_4.
\end{aligned}$$

Example. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic equation, after a little computation, is

$$(\lambda^2 + 1)^2 = 0.$$

Hence A has eigenvalues $\pm i$, each repeated twice.

Solving the system $(A - iI)X = 0$ yields one linearly independent complex eigenvector $V_1 = (1, 1 - i, 0, 0)$ associated to i. Then $\overline{V_1}$ is an eigenvector associated to the eigenvalue $-i$.

Next we solve the system $(A - iI)X = V_1$ to find $V_2 = (0, 0, 1 - i, 1)$. Then $\overline{V_2}$ solves the system $(A - iI)X = \overline{V_1}$. Finally, choose

$$\begin{aligned} W_1 &= \left(V_1 + \overline{V_1}\right)/2 = \text{Re } V_1 \\ W_2 &= -i\left(V_1 - \overline{V_1}\right)/2 = \text{Im } V_1 \\ W_3 &= \left(V_2 + \overline{V_2}\right)/2 = \text{Re } V_2 \\ W_4 &= -i\left(V_2 - \overline{V_2}\right)/2 = \text{Im } V_2 \end{aligned}$$

and let $TE_j = W_j$ for $j = 1, \ldots, 4$. We have

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

and we find the canonical form

$$T^{-1}AT = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

■

Example. Let

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

The characteristic equation for A is

$$(2-\lambda)^2((2-\lambda)^2+1)=0$$

so the eigenvalues are $2 \pm i$ and 2 (with multiplicity 2).

Solving the equations $(A-(2+i)I)X = 0$ yields an eigenvector $V = (0,-i,0,1)$ for $2+i$. Let $W_1 = (0,0,0,1)$ and $W_2 = (0,-1,0,0)$ be the real and imaginary parts of V.

Solving the equations $(A-2I)X = 0$ yields only one eigenvector associated to 2, namely, $W_3 = (1,0,0,0)$. Then we solve $(A-2I)X = W_3$ to find $W_4 = (0,0,1,0)$. Setting $TE_j = W_j$ as usual puts A into the canonical form

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

as is easily checked. ∎

5.6 Genericity

We have mentioned several times that "most" matrices have distinct eigenvalues. Our goal in this section is to make this precise.

Recall that a set $\mathcal{U} \subset \mathbb{R}^n$ is *open* if whenever $X \in \mathcal{U}$ there is an open ball about X contained in $\mathcal{U}$; that is, for some $a > 0$ (depending on X) the open ball about X of radius a,

$$\{Y \in \mathbb{R}^n \mid |Y - X| < a\},$$

is contained in $\mathcal{U}$. Using geometrical language we say that if X belongs to an open set $\mathcal{U}$, any point sufficiently near to X also belongs to $\mathcal{U}$.

Another kind of subset of $\mathbb{R}^n$ is a *dense* set: $\mathcal{U} \subset \mathbb{R}^n$ is dense if there are points in $\mathcal{U}$ arbitrarily close to each point in $\mathbb{R}^n$. More precisely, if $X \in \mathbb{R}^n$, then for every $\epsilon > 0$ there exists some $Y \in \mathcal{U}$ with $|X - Y| < \epsilon$. Equivalently, $\mathcal{U}$ is dense in $\mathbb{R}^n$ if $\mathcal{V} \cap \mathcal{U}$ is nonempty for every nonempty open set $\mathcal{V} \subset \mathbb{R}^n$. For example, the rational numbers form a dense subset of $\mathbb{R}$, as do the irrational numbers. Similarly

$$\{(x,y) \in \mathbb{R}^2 \mid \text{both } x \text{ and } y \text{ are rational}\}$$

is a dense subset of the plane.

An interesting kind of subset of $\mathbb{R}^n$ is a set that is both open and dense. Such a set $\mathcal{U}$ is characterized by the following properties: Every point in the complement of $\mathcal{U}$ can be approximated arbitrarily closely by points of $\mathcal{U}$ (since $\mathcal{U}$ is dense); but no point in $\mathcal{U}$ can be approximated arbitrarily closely by points in the complement (because $\mathcal{U}$ is open).

Here is a simple example of an open and dense subset of $\mathbb{R}^2$:

$$\mathcal{V} = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 1\}.$$

This, of course, is the complement in $\mathbb{R}^2$ of the hyperbola defined by $xy = 1$. Suppose $(x_0, y_0) \in \mathcal{V}$. Then $x_0 y_0 \neq 1$ and if $|x - x_0|$, $|y - y_0|$ are small enough, then $xy \neq 1$; this proves that $\mathcal{V}$ is open. Given any $(x_0, y_0) \in \mathbb{R}^2$, we can find (x, y) as close as we like to (x_0, y_0) with $xy \neq 1$; this proves that $\mathcal{V}$ is dense.

An open and dense set is a very fat set, as the following proposition shows:

Proposition. *Let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ be open and dense subsets of $\mathbb{R}^n$. Then*

$$\mathcal{V} = \mathcal{V}_1 \cap \ldots \cap \mathcal{V}_m$$

is also open and dense.

Proof: It can be easily shown that the intersection of a finite number of open sets is open, so $\mathcal{V}$ is open. To prove that $\mathcal{V}$ is dense let $\mathcal{U} \subset \mathbb{R}^n$ be a nonempty open set. Then $\mathcal{U} \cap \mathcal{V}_1$ is nonempty since $\mathcal{V}_1$ is dense. Because $\mathcal{U}$ and $\mathcal{V}_1$ are open, $\mathcal{U} \cap \mathcal{V}_1$ is also open. Since $\mathcal{U} \cap \mathcal{V}_1$ is open and nonempty, $(\mathcal{U} \cap \mathcal{V}_1) \cap \mathcal{V}_2$ is nonempty because $\mathcal{V}_2$ is dense. Since $\mathcal{V}_1$ is open, $\mathcal{U} \cap \mathcal{V}_1 \cap \mathcal{V}_2$ is open. Thus $(\mathcal{U} \cap \mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{V}_3$ is nonempty, and so on. So $\mathcal{U} \cap \mathcal{V}$ is nonempty, which proves that $\mathcal{V}$ is dense in $\mathbb{R}^n$. ■

We therefore think of a subset of $\mathbb{R}^n$ as being large if this set contains an open and dense subset. To make precise what we mean by "most" matrices, we need to transfer the notion of an open and dense set to the set of all matrices.

Let $L(\mathbb{R}^n)$ denote the set of $n \times n$ matrices, or, equivalently, the set of linear maps of $\mathbb{R}^n$. To discuss open and dense sets in $L(\mathbb{R}^n)$, we need to have a notion of how far apart two given matrices in $L(\mathbb{R}^n)$ are. But we can do this by simply writing all of the entries of a matrix as one long vector (in a specified order) and thereby thinking of $L(\mathbb{R}^n)$ as $\mathbb{R}^{n^2}$.

Theorem. *The set $\mathcal{M}$ of matrices in $L(\mathbb{R}^n)$ that have n distinct eigenvalues is open and dense in $L(\mathbb{R}^n)$.*

Proof: We first prove that $\mathcal{M}$ is dense. Let $A \in L(\mathbb{R}^n)$. Suppose that A has some repeated eigenvalues. The proposition from the previous section states that we can find a matrix T such that $T^{-1}AT$ assumes one of two forms. Either we have a canonical form with blocks along the diagonal of the form

$$\text{(i)} \quad \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \quad \text{or} \quad \text{(ii)} \quad \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix}$$

where $\alpha, \beta, \lambda \in \mathbb{R}$ with $\beta \neq 0$ and

$$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or else we have a pair of separate diagonal blocks (λ) or C_2. Either case can be handled as follows.

Choose distinct values λ_j such that $|\lambda - \lambda_j|$ is as small as desired, and replace block (i) above by

$$\begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_j \end{pmatrix}.$$

This new block now has distinct eigenvalues. In block (ii) we may similarly replace each 2×2 block

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with distinct α_i's. The new matrix thus has distinct eigenvalues $\alpha_i \pm \beta$. In this fashion, we find a new matrix B arbitrarily close to $T^{-1}AT$ with distinct eigenvalues. Then the matrix TBT^{-1} also has distinct eigenvalues and, moreover, this matrix is arbitrarily close to A. Indeed, the function $F\colon L(\mathbb{R}^n) \to L(\mathbb{R}^n)$ given by $F(M) = TMT^{-1}$ where T is a fixed invertible matrix is a continuous function on $L(\mathbb{R}^n)$ and hence takes matrices close to $T^{-1}AT$ to new matrices close to A. This shows that $\mathcal{M}$ is dense.

To prove that $\mathcal{M}$ is open, consider the characteristic polynomial of a matrix $A \in L(\mathbb{R}^n)$. If we vary the entries of A slightly, then the characteristic polynomial's coefficients vary only slightly. Hence the roots of this polynomial in $\mathbb{C}$

move only slightly as well. Thus, if we begin with a matrix whose eigenvalues are distinct, nearby matrices have this property as well. This proves that $\mathcal{M}$ is open. ∎

A property $\mathcal{P}$ of matrices is a *generic property* if the set of matrices having property $\mathcal{P}$ contains an open and dense set in $L(\mathbb{R}^n)$. Thus a property is generic if it is shared by some open and dense set of matrices (and perhaps other matrices as well). Intuitively speaking, a generic property is one that "almost all" matrices have. Thus, having all distinct eigenvalues is a generic property of $n \times n$ matrices.

EXERCISES

1. Prove that the determinant of a 3×3 matrix can be computed by expanding along any row or column.

2. Find the eigenvalues and eigenvectors of the following matrices:

$$\text{(a)}\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{(b)}\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \quad \text{(c)}\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\text{(d)}\ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix} \quad \text{(e)}\ \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & -2 & -1 & -4 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

3. Describe the regions in a, b, c–space where the matrix

$$\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$$

has real, complex, and repeated eigenvalues.

4. Describe the regions in a, b, c–space where the matrix

$$\begin{pmatrix} a & 0 & 0 & a \\ 0 & a & b & 0 \\ 0 & c & a & 0 \\ a & 0 & 0 & a \end{pmatrix}$$

has real, complex, and repeated eigenvalues.

5. Put the following matrices in canonical form

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

6. Suppose that a 5×5 matrix has eigenvalues 2 and $1 \pm i$. List all possible canonical forms for a matrix of this type.

7. Let L be the elementary matrix that interchanges the ith and jth rows of a given matrix. That is, L has 1's along the diagonal, with the exception that $\ell_{ii} = \ell_{jj} = 0$ but $\ell_{ij} = \ell_{ji} = 1$. Prove that $\det L = -1$.

8. Find a basis for both Ker T and Range T when T is the matrix

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 9 & 6 \\ 1 & 4 & 1 \\ 2 & 7 & 1 \end{pmatrix}$

9. Suppose A is a 4×4 matrix that has a single real eigenvalue λ and only one independent eigenvector. Prove that A may be put in canonical form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

10. Suppose A is a 4×4 matrix with a single real eigenvalue and two linearly independent eigenvectors. Describe the possible canonical forms for A and show that A may indeed be transformed into one of these canonical forms. Describe explicitly the conditions under which A is transformed into a particular form.

11. Show that if A and/or B are noninvertible matrices, then AB is also noninvertible.

12. Suppose that $\mathcal{S}$ is a subset of $\mathbb{R}^n$ having the following properties:

(a) If $X, Y \in \mathcal{S}$, then $X + Y \in \mathcal{S}$;

(b) If $X \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $\alpha X \in \mathcal{S}$.

Prove that $\mathcal{S}$ may be written as the collection of all possible linear combinations of a finite set of vectors.

13. Which of the following subsets of $\mathbb{R}^n$ are open and/or dense? Give a brief reason in each case.

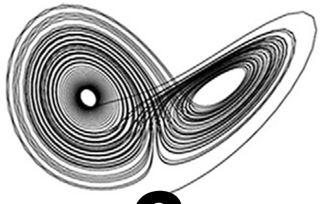
(a) $\mathcal{U}_1 = \{(x, y) \mid y > 0\}$;
(b) $\mathcal{U}_2 = \{(x, y) \mid x^2 + y^2 \neq 1\}$;
(c) $\mathcal{U}_3 = \{(x, y) \mid x \text{ is irrational}\}$;
(d) $\mathcal{U}_4 = \{(x, y) \mid x \text{ and } y \text{ are not integers}\}$;
(e) $\mathcal{U}_5$ is the complement of a set C_1 where C_1 is closed and not dense;
(f) $\mathcal{U}_6$ is the complement of a set C_2 which contains exactly 6 billion and two distinct points.

14. Each of the following properties defines a subset of real $n \times n$ matrices. Which of these sets are open and/or dense in the $L(\mathbb{R}^n)$? Give a brief reason in each case.

(a) $\det A \neq 0$.
(b) Trace A is rational.
(c) Entries of A are not integers.
(d) $3 \leq \det A < 4$.
(e) $-1 < |\lambda| < 1$ for every eigenvalue λ.
(f) A has no real eigenvalues.
(g) Each real eigenvalue of A has multiplicity one.

15. Which of the following properties of linear maps on $\mathbb{R}^n$ are generic?

(a) $|\lambda| \neq 1$ for every eigenvalue λ.
(b) $n = 2$; some eigenvalue is not real.
(c) $n = 3$; some eigenvalue is not real.
(d) No solution of $X' = AX$ is periodic (except the zero solution).
(e) There are n distinct eigenvalues, each with distinct imaginary parts.
(f) $AX \neq X$ and $AX \neq -X$ for all $X \neq 0$.



6
Higher Dimensional Linear Systems

After our little sojourn into the world of linear algebra, it's time to return to differential equations and, in particular, to the task of solving higher dimensional linear systems with constant coefficients. As in the linear algebra chapter, we have to deal with a number of different cases.

6.1 Distinct Eigenvalues

Consider first a linear system $X' = AX$ where the $n \times n$ matrix A has n distinct, real eigenvalues $\lambda_1, \ldots, \lambda_n$. By the results in Chapter 5, there is a change of coordinates T so that the new system $Y' = (T^{-1}AT)Y$ assumes the particularly simple form

$$\begin{aligned} y_1' &= \lambda_1 y_1 \\ &\vdots \\ y_n' &= \lambda_n y_n. \end{aligned}$$

The linear map T is the map that takes the standard basis vector E_j to the eigenvector V_j associated to λ_j. Clearly, a function of the form

$$Y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

is a solution of $Y' = (T^{-1}AT)Y$ that satisfies the initial condition $Y(0) = (c_1, \ldots, c_n)$. As in Chapter 3, this is the only such solution, because if

$$W(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}$$

is another solution, then differentiating each expression $w_j(t)\exp(-\lambda_j t)$, we find

$$\frac{d}{dt} w_j(t)e^{-\lambda_j t} = (w_j' - \lambda_j w_j)e^{-\lambda_j t} = 0.$$

Hence $w_j(t) = c_j \exp(\lambda_j t)$ for each j. Therefore the collection of solutions $Y(t)$ yields the general solution of $Y' = (T^{-1}AT)Y$.

It then follows that $X(t) = TY(t)$ is the general solution of $X' = AX$, so this general solution may be written in the form

$$X(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} V_j.$$

Now suppose that the eigenvalues $\lambda_1, \ldots, \lambda_k$ of A are negative, while the eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ are positive. Since there are no zero eigenvalues, the system is hyperbolic. Then any solution that starts in the subspace spanned by the vectors $V_1, \ldots, V_k$ must first of all stay in that subspace for all time since $c_{k+1} = \cdots = c_n = 0$. Secondly, each such solution tends to the origin as $t \to \infty$. In analogy with the terminology introduced for planar systems, we call this subspace the *stable subspace.* Similarly, the subspace spanned by $V_{k+1}, \ldots, V_n$ contains solutions that move away from the origin. This subspace is the *unstable subspace.* All other solutions tend toward the stable subspace as time goes backward and toward the unstable subspace as time increases. Therefore this system is a higher dimensional analog of a *saddle.*

Example. Consider

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X.$$

In Section 5.2 in Chapter 5, we showed that this matrix has eigenvalues 2, 1, and -1 with associated eigenvectors $(3, 2, 1)$, $(1, 0, 0)$, and $(0, 1, 2)$, respectively. Therefore the matrix

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

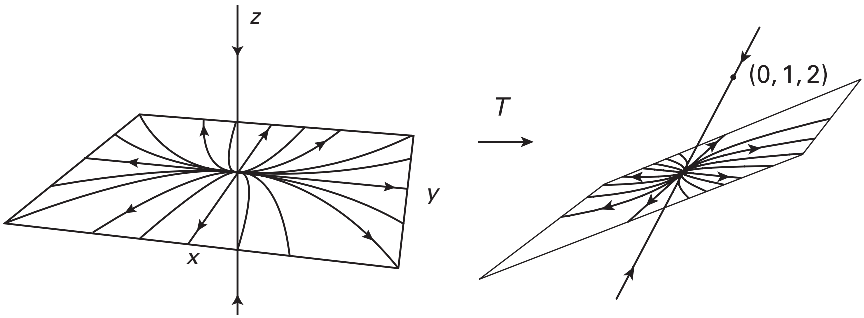


Figure 6.1 The stable and unstable subspaces of a saddle in dimension 3. On the left, the system is in canonical form.

converts $X' = AX$ to

$$Y' = (T^{-1}AT)Y = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Y,$$

which we can solve immediately. Multiplying the solution by T then yields the general solution

$$X(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

of $X' = AX$. The straight line through the origin and $(0, 1, 2)$ is the stable line, while the plane spanned by $(3, 2, 1)$ and $(1, 0, 0)$ is the unstable plane. A collection of solutions of this system as well as the system $Y' = (T^{-1}AT)Y$ is displayed in Figure 6.1. ■

Example. If the 3×3 matrix A has three real, distinct eigenvalues that are negative, then we may find a change of coordinates so that the system assumes the form

$$Y' = (T^{-1}AT)Y = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} Y$$

where $\lambda_3 < \lambda_2 < \lambda_1 < 0$. All solutions therefore tend to the origin and so we have a higher dimensional *sink*. See Figure 6.2. For an initial condition (x_0, y_0, z_0) with all three coordinates nonzero, the corresponding solution tends to the origin tangentially to the x-axis (see Exercise 2 at the end of the chapter). ■

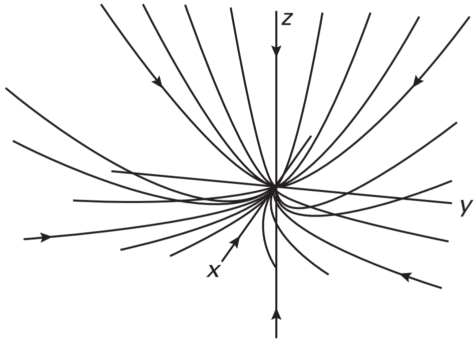


Figure 6.2 A sink in three dimensions.

Now suppose that the $n \times n$ matrix A has n distinct eigenvalues, of which k_1 are real and $2k_2$ are nonreal, so that $n = k_1 + 2k_2$. Then, as in Chapter 5, we may change coordinates so that the system assumes the form

$$\begin{aligned} x_j' &= \lambda_j x_j \\ u_\ell' &= \alpha_\ell u_\ell + \beta_\ell v_\ell \\ v_\ell' &= -\beta_\ell u_\ell + \alpha_\ell v_\ell \end{aligned}$$

for $j = 1, \ldots, k_1$ and $\ell = 1, \ldots, k_2$. As in Chapter 3, we therefore have solutions of the form

$$\begin{aligned} x_j(t) &= c_j e^{\lambda_j t} \\ u_\ell(t) &= p_\ell e^{\alpha_\ell t} \cos \beta_\ell t + q_\ell e^{\alpha_\ell t} \sin \beta_\ell t \\ v_\ell(t) &= -p_\ell e^{\alpha_\ell t} \sin \beta_\ell t + q_\ell e^{\alpha_\ell t} \cos \beta_\ell t. \end{aligned}$$

As before, it is straightforward to check that this is the general solution. We have therefore shown:

Theorem. *Consider the system $X' = AX$ where A has distinct eigenvalues $\lambda_1, \ldots, \lambda_{k_1} \in \mathbb{R}$ and $\alpha_1 + i\beta_1, \ldots, \alpha_{k_2} + i\beta_{k_2} \in \mathbb{C}$. Let T be the matrix that puts A in the canonical form*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_{k_1} & & & \\ & & & B_1 & & \\ & & & & \ddots & \\ & & & & & B_{k_2} \end{pmatrix}$$

where

$$B_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}.$$

Then the general solution of $X' = AX$ is $TY(t)$ where

$$Y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_k e^{\lambda_{k_1} t} \\ a_1 e^{\alpha_1 t} \cos \beta_1 t + b_1 e^{\alpha_1 t} \sin \beta_1 t \\ -a_1 e^{\alpha_1 t} \sin \beta_1 t + b_1 e^{\alpha_1 t} \cos \beta_1 t \\ \vdots \\ a_{k_2} e^{\alpha_{k_2} t} \cos \beta_{k_2} t + b_{k_2} e^{\alpha_{k_2} t} \sin \beta_{k_2} t \\ -a_{k_2} e^{\alpha_{k_2} t} \sin \beta_{k_2} t + b_{k_2} e^{\alpha_{k_2} t} \cos \beta_{k_2} t \end{pmatrix}$$ ■

As usual, the columns of the matrix T in this theorem are the eigenvectors (or the real and imaginary parts of the eigenvectors) corresponding to each eigenvalue. Also, as before, the subspace spanned by the eigenvectors corresponding to eigenvalues with negative (resp., positive) real parts is the stable (resp., unstable) subspace.

Example. Consider the system

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

whose matrix is already in canonical form. The eigenvalues are $\pm i, -1$. The solution satisfying the initial condition (x_0, y_0, z_0) is given by

$$Y(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so this is the general solution. The phase portrait for this system is displayed in Figure 6.3. The stable line lies along the z-axis, whereas all solutions in the xy–plane travel around circles centered at the origin. In fact, each solution that does not lie on the stable line actually lies on a cylinder in $\mathbb{R}^3$ given by $x^2 + y^2 = \text{constant}$. These solutions spiral toward the circular solution of radius $\sqrt{x_0^2 + y_0^2}$ in the xy–plane if $z_0 \neq 0$. ■

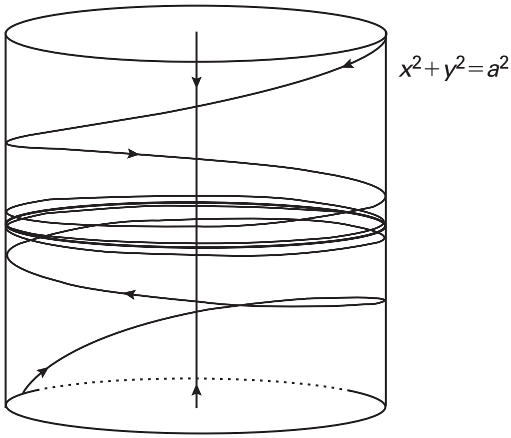


Figure 6.3 The phase portrait for a spiral center.

Example. Now consider $X' = AX$ where

$$A = \begin{pmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{pmatrix}.$$

The characteristic equation is

$$-\lambda^3 + 0.8\lambda^2 - 0.81\lambda + 1.01 = 0,$$

which we have kindly factored for you into

$$(1-\lambda)(\lambda^2 + 0.2\lambda + 1.01) = 0.$$

Therefore the eigenvalues are the roots of this equation, which are 1 and $-0.1 \pm i$. Solving $(A - (-0.1 + i)I)X = 0$ yields the eigenvector $(-i, 1, 1)$ associated to $-0.1 + i$. Let $V_1 = \text{Re}\,(-i, 1, 1) = (0, 1, 1)$ and $V_2 = \text{Im}\,(-i, 1, 1) = (-1, 0, 0)$. Solving $(A - I)X = 0$ yields $V_3 = (0, 1, 0)$ as an eigenvector corresponding to $\lambda = 1$. Then the matrix whose columns are the V_i,

$$T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

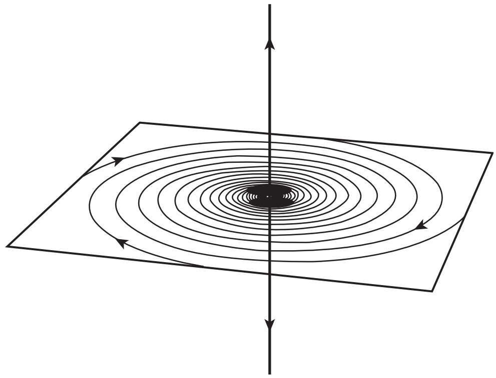


Figure 6.4 A spiral saddle in canonical form.

converts $X' = AX$ into

$$Y' = \begin{pmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y.$$

This system has an unstable line along the z-axis, while the xy–plane is the stable plane. Note that solutions spiral into 0 in the stable plane. We call this system a *spiral saddle* (see Figure 6.4). Typical solutions of the stable plane spiral toward the z-axis while the z-coordinate meanwhile increases or decreases (see Figure 6.5). ■

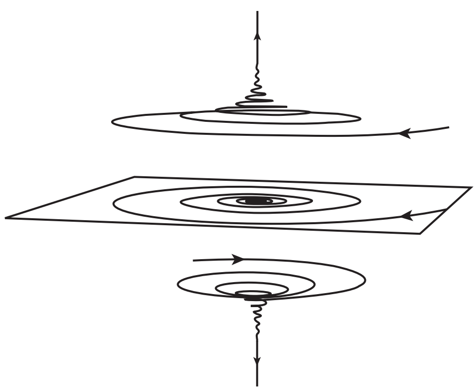


Figure 6.5 Typical solutions of the spiral saddle tend to spiral toward the unstable line.

6.2 Harmonic Oscillators

Consider a pair of undamped harmonic oscillators whose equations are

$$\begin{aligned} x_1'' &= -\omega_1^2 x_1 \\ x_2'' &= -\omega_2^2 x_2. \end{aligned}$$

We can almost solve these equations by inspection as visions of $\sin \omega t$ and $\cos \omega t$ pass through our minds. But let's push on a bit, first to illustrate the theorem in the previous section in the case of nonreal eigenvalues, but more importantly to introduce some interesting geometry.

We first introduce the new variables $y_j = x_j'$ for $j = 1, 2$ so that the equations can be written as a system

$$\begin{aligned} x_j' &= y_j \\ y_j' &= -\omega_j^2 x_j. \end{aligned}$$

In matrix form, this system is $X' = AX$ where $X = (x_1, y_1, x_2, y_2)$ and

$$A = \begin{pmatrix} 0 & 1 & & \\ -\omega_1^2 & 0 & & \\ & & 0 & 1 \\ & & -\omega_2^2 & 0 \end{pmatrix}.$$

This system has eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$. An eigenvector corresponding to $i\omega_1$ is $V_1 = (1, i\omega_1, 0, 0)$ while $V_2 = (0, 0, 1, i\omega_2)$ is associated to $i\omega_2$. Let W_1 and W_2 be the real and imaginary parts of V_1, and let W_3 and W_4 be the same for V_2. Then, as usual, we let $TE_j = W_j$ and the linear map T puts this system into canonical form with the matrix

$$T^{-1}AT = \begin{pmatrix} 0 & \omega_1 & & \\ -\omega_1 & 0 & & \\ & & 0 & \omega_2 \\ & & -\omega_2 & 0 \end{pmatrix}.$$

We then derive the general solution

$$Y(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ -a_1 \sin \omega_1 t + b_1 \cos \omega_1 t \\ a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \\ -a_2 \sin \omega_2 t + b_2 \cos \omega_2 t \end{pmatrix}$$

just as we originally expected.

We could say that this is the end of the story and stop here since we have the formulas for the solution. However, let's push on a bit more.

Each pair of solutions $(x_j(t), y_j(t))$ for $j = 1, 2$ is clearly a periodic solution of the equation with period $2\pi/\omega_j$, but this does not mean that the full four-dimensional solution is a periodic function. Indeed, the full solution is a periodic function with period τ if and only if there exist integers m and n such that

$$\omega_1\tau = m \cdot 2\pi \quad \text{and} \quad \omega_2\tau = n \cdot 2\pi.$$

Thus, for periodicity, we must have

$$\tau = \frac{2\pi m}{\omega_1} = \frac{2\pi n}{\omega_2}$$

or, equivalently,

$$\frac{\omega_2}{\omega_1} = \frac{n}{m}.$$

That is, the ratio of the two frequencies of the oscillators must be a rational number. In Figure 6.6 we have plotted $(x_1(t), x_2(t))$ for particular solution of this system when the ratio of the frequencies is 5/2.

When the ratio of the frequencies is irrational, something very different happens. To understand this, we make another (and much more familiar) change of coordinates. In canonical form, our system currently is

$$x_j' = \omega_j y_j$$
$$y_j' = -\omega_j x_j.$$

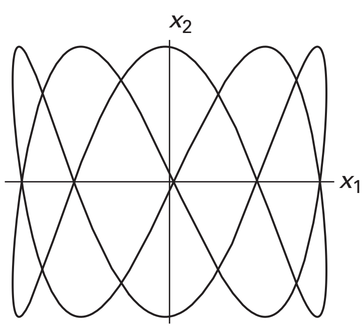


Figure 6.6 A solution with frequency ratio 5/2 projected into the $x_1 x_2$–plane. Note that $x_2(t)$ oscillates five times and $x_1(t)$ only twice before returning to the initial position.

Let's now introduce polar coordinates (r_j, θ_j) in place of the x_j and y_j variables. Differentiating

$$r_j^2 = x_j^2 + y_j^2,$$

we find

$$\begin{aligned} 2r_j r_j' &= 2x_j x_j' + 2y_j y_j' \\ &= 2x_j y_j \omega_j - 2x_j y_j \omega_j \\ &= 0. \end{aligned}$$

Therefore $r_j' = 0$ for each j. Also, differentiating the equation

$$\tan \theta_j = \frac{y_j}{x_j}$$

yields

$$\begin{aligned} (\sec^2 \theta_j) \theta_j' &= \frac{y_j' x_j - y_j x_j'}{x_j^2} \\ &= \frac{-\omega_j r_j^2}{r_j^2 \cos^2 \theta_j} \end{aligned}$$

from which we find

$$\theta_j' = -\omega_j.$$

So, in polar coordinates, these equations really are quite simple:

$$\begin{aligned} r_j' &= 0 \\ \theta_j' &= -\omega_j. \end{aligned}$$

The first equation tells us that both r_1 and r_2 remain constant along any solution. Then, no matter what we pick for our initial r_1 and r_2 values, the θ_j equations remain the same. Hence we may as well restrict our attention to $r_1 = r_2 = 1$. The resulting set of points in $\mathbb{R}^4$ is a *torus*—the surface of a doughnut—although this is a little difficult to visualize in four-dimensional space. However, we know that we have two independent variables on this set, namely, θ_1 and θ_2, and both are periodic with period 2π. So this is akin to the two independent circular directions that parameterize the familiar torus in $\mathbb{R}^3$.

Restricted to this torus, the equations now read

$$\theta_1' = -\omega_1$$
$$\theta_2' = -\omega_2.$$

It is convenient to think of θ_1 and θ_2 as variables in a square of sidelength 2π where we glue together the opposite sides $\theta_j = 0$ and $\theta_j = 2\pi$ to make the torus. In this square our vector field now has constant slope

$$\frac{\theta_2'}{\theta_1'} = \frac{\omega_2}{\omega_1}.$$

Therefore solutions lie along straight lines with slope ω_2/ω_1 in this square. When a solution reaches the edge $\theta_1 = 2\pi$ (say, at $\theta_2 = c$), it instantly reappears on the edge $\theta_1 = 0$ with the θ_2 coordinate given by c, and then continues onward with slope ω_2/ω_1. A similar identification occurs when the solution meets $\theta_2 = 2\pi$.

So now we have a simplified geometric vision of what happens to these solutions on these tori. But what really happens? The answer depends on the ratio ω_2/ω_1. If this ratio is a rational number, say, n/m, then the solution starting at $(\theta_1(0), \theta_2(0))$ will pass horizontally through the torus exactly m times and vertically n times before returning to its starting point. This is the periodic solution we observed above. Incidentally, the picture of the straight line solutions in the $\theta_1\theta_2$–plane is not at all the same as our depiction of solutions in the x_1x_2–plane as shown in Figure 6.6.

In the irrational case, something quite different occurs. See Figure 6.7. To understand what is happening here, we return to the notion of a *Poincaré map* discussed in Chapter 1. Consider the circle $\theta_1 = 0$, the left-hand edge of our square representation of the torus. Given an initial point on this circle, say, $\theta_2 = x_0$, we follow the solution starting at this point until it next hits $\theta_1 = 2\pi$.

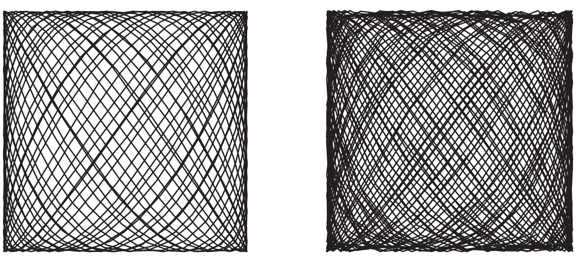


Figure 6.7 A solution with frequency ratio $\sqrt{2}$ projected into the x_1x_2–plane, the left curve computed up to time 50π, the right to time 100π.

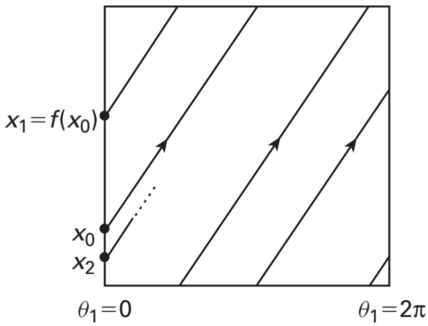


Figure 6.8 The Poincaré map on the circle $\theta_1 = 0$ in the $\theta_1\theta_2$ torus.

By our identification, this solution has now returned to the circle $\theta_1 = 0$. The solution may cross the boundary $\theta_2 = 2\pi$ several times in making this transit, but it does eventually return to $\theta_1 = 0$. So we may define the Poincaré map on $\theta_1 = 0$ by assigning to x_0 on this circle the corresponding coordinate of the point of first return. Suppose that this first return occurs at the point $\theta_2(\tau)$ where τ is the time for which $\theta_1(\tau) = 2\pi$. Since $\theta_1(t) = \theta_1(0) + \omega_1 t$, we have $\tau = 2\pi/\omega_1$. Hence $\theta_2(\tau) = x_0 + \omega_2(2\pi/\omega_1)$. Therefore the Poincaré map on the circle may be written as

$$f(x_0) = x_0 + 2\pi(\omega_2/\omega_1) \bmod 2\pi$$

where $x_0 = \theta_2(0)$ is our initial θ_2 coordinate on the circle. See Figure 6.8. Thus the Poincaré map on the circle is just the function that rotates points on the circle by angle $2\pi(\omega_2/\omega_1)$. Since ω_2/ω_1 is irrational, this function is called an *irrational rotation* of the circle.

Definition

The set of points $x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \ldots, x_n = f(x_{n-1})$ is called the *orbit* of x_0 under iteration of f.

The orbit of x_0 tracks how our solution successively crosses $\theta_1 = 2\pi$ as time increases.

Proposition. *Suppose ω_2/ω_1 is irrational. Then the orbit of any initial point x_0 on the circle $\theta_1 = 0$ is dense in the circle.*

Proof: Recall from Section 5.6 that a subset of the circle is *dense* if there are points in this subset that are arbitrarily close to any point whatsoever in the circle. Therefore we must show that, given any point z on the circle and any $\epsilon > 0$, there is a point x_n on the orbit of x_0 such that $|z - x_n| < \epsilon$ where z and x_n are measured mod 2π. To see this, observe first that there must be n, m for which $m > n$ and $|x_n - x_m| < \epsilon$. Indeed, we know that the orbit of x_0 is not a finite set of points since ω_2/ω_1 is irrational. Hence there must be at least two of these points whose distance apart is less than ϵ since the circle has finite circumference. These are the points x_n and x_m (actually, there must be infinitely many such points). Now rotate these points in the reverse direction exactly n times. The points x_n and x_m are rotated to x_0 and x_{m-n}, respectively. We find, after this rotation, that $|x_0 - x_{m-n}| < \epsilon$. Now x_{m-n} is given by rotating the circle through angle $(m-n)2\pi(\omega_2/\omega_1)$, which, mod 2π, is therefore a rotation of angle less than ϵ. Hence, performing this rotation again, we find

$$|x_{2(m-n)} - x_{m-n}| < \epsilon$$

as well, and, inductively,

$$|x_{k(m-n)} - x_{(k-1)(m-n)}| < \epsilon$$

for each k. Thus we have found a sequence of points obtained by repeated rotation through angle $(m - n)2\pi(\omega_2/\omega_1)$, and each of these points is within ϵ of its predecessor. Hence there must be a point of this form within ϵ of z. ■

Since the orbit of x_0 is dense in the circle $\theta_1 = 0$, it follows that the straight-line solutions connecting these points in the square are also dense, and so the original solutions are dense in the torus on which they reside. This accounts for the densely packed solution shown projected into the x_1x_2–plane in Figure 6.7 when $\omega_2/\omega_1 = \sqrt{2}$.

Returning to the actual motion of the oscillators, we see that when ω_2/ω_1 is irrational, the masses do not move in periodic fashion. However, they do come back very close to their initial positions over and over again as time goes on due to the density of these solutions on the torus. These types of motions are called *quasi-periodic motions.* In Exercise 7 we investigate a related set of equations, namely, a pair of coupled oscillators.

6.3 Repeated Eigenvalues

As we saw in the previous chapter, the solution of systems with repeated real eigenvalues reduces to solving systems whose matrices contain blocks of the form

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

Example. Let

$$X' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} X.$$

The only eigenvalue for this system is λ, and its only eigenvector is $(1, 0, 0)$. We may solve this system as we did in Chapter 3, by first noting that $x_3' = \lambda x_3$, so we must have

$$x_3(t) = c_3 e^{\lambda t}.$$

Now we must have

$$x_2' = \lambda x_2 + c_3 e^{\lambda t}.$$

As in Chapter 3, we guess a solution of the form

$$x_2(t) = c_2 e^{\lambda t} + \alpha t e^{\lambda t}.$$

Substituting this guess into the differential equation for x_2', we determine that $\alpha = c_3$ and find

$$x_2(t) = c_2 e^{\lambda t} + c_3 t e^{\lambda t}.$$

Finally, the equation

$$x_1' = \lambda x_1 + c_2 e^{\lambda t} + c_3 t e^{\lambda t}$$

suggests the guess

$$x_1(t) = c_1 e^{\lambda t} + \alpha t e^{\lambda t} + \beta t^2 e^{\lambda t}.$$

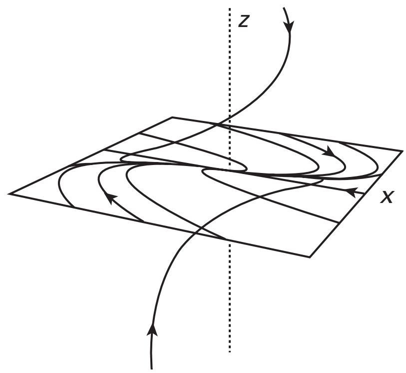


Figure 6.9 The phase portrait for repeated real eigenvalues.

Solving as above, we find

$$x_1(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} + c_3 \frac{t^2}{2} e^{\lambda t}.$$

Altogether, we find

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix},$$

which is the general solution. Despite the presence of the polynomial terms in this solution, when $\lambda < 0$, the exponential term dominates and all solutions do tend to zero. Some representative solutions when $\lambda < 0$ are shown in Figure 6.9. Note that there is only one straight-line solution for this system; this solution lies on the x-axis. Also, the xy–plane is invariant and solutions there behave exactly as in the planar repeated eigenvalue case. ■

Example. Consider the following four-dimensional system:

$$\begin{aligned} x_1' &= x_1 + x_2 - x_3 \\ x_2' &= x_2 + x_4 \\ x_3' &= x_3 + x_4 \\ x_4' &= x_4 \end{aligned}$$

We may write this system in matrix form as

$$X' = AX = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} X.$$

Because A is upper triangular, all of the eigenvalues are equal to 1. Solving $(A - I)X = 0$, we find two independent eigenvectors $V_1 = (1, 0, 0, 0)$ and $W_1 = (0, 1, 1, 0)$. This reduces the possible canonical forms for A to two possibilities. Solving $(A - I)X = V_1$ yields one solution $V_2 = (0, 1, 0, 0)$, and solving $(A - I)X = W_1$ yields another solution $W_2 = (0, 0, 0, 1)$. Thus we know that the system $X' = AX$ may be tranformed into

$$Y' = (T^{-1}AT)Y = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} Y$$

where the matrix T is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solutions of $Y' = (T^{-1}AT)Y$ therefore are given by

$$\begin{aligned} y_1(t) &= c_1 e^t + c_2 te^t \\ y_2(t) &= c_2 e^t \\ y_3(t) &= c_3 e^t + c_4 te^t \\ y_4(t) &= c_4 e^t. \end{aligned}$$

Applying the change of coordinates T, we find the general solution of the original system

$$\begin{aligned} x_1(t) &= c_1 e^t + c_2 te^t \\ x_2(t) &= c_2 e^t + c_3 e^t + c_4 te^t \\ x_3(t) &= c_3 e^t + c_4 te^t \\ x_4(t) &= c_4 e^t. \end{aligned}$$

■

6.4 The Exponential of a Matrix

We turn now to an alternative and elegant approach to solving linear systems using the exponential of a matrix. In a certain sense, this is the more natural way to attack these systems.

Recall from Section 1.1 in Chapter 1 how we solved the 1×1 "system" of linear equations $x' = ax$ where our matrix was now simply (a). We did not go through the process of finding eigenvalues and eigenvectors here. (Well, actually, we did, but the process was pretty simple.) Rather, we just exponentiated the matrix (a) to find the general solution $x(t) = c\exp(at)$. In fact, this process works in the general case where A is $n \times n$. All we need to know is how to exponentiate a matrix.

Here's how: Recall from calculus that the exponential function can be expressed as the infinite series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

We know that this series converges for every $x \in \mathbb{R}$. Now we can add matrices, we can raise them to the power k, and we can multiply each entry by $1/k!$. So this suggests that we can use this series to exponentiate them as well.

Definition
Let A be an $n \times n$ matrix. We define the *exponential* of A to be the matrix given by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Of course we have to worry about what it means for this sum of matrices to converge, but let's put that off and try to compute a few examples first.

Example. Let

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Then we have

$$A^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}$$

so that

$$\exp(A) = \begin{pmatrix} \sum_{k=0}^{\infty} \lambda^k/k! & 0 \\ 0 & \sum_{k=0}^{\infty} \mu^k/k! \end{pmatrix} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}$$

as you may have guessed. ■

Example. For a slightly more complicated example, let

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

We compute

$$A^0 = I, \quad A^2 = -\beta^2 I, \quad A^3 = -\beta^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$A^4 = \beta^4 I, \quad A^5 = \beta^5 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \ldots$$

so we find

$$\exp(A) = \begin{pmatrix} \sum_{k=0}^{\infty} (-1)^k \dfrac{\beta^{2k}}{(2k)!} & \sum_{k=0}^{\infty} (-1)^k \dfrac{\beta^{2k+1}}{(2k+1)!} \\ -\sum_{k=0}^{\infty} (-1)^k \dfrac{\beta^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} (-1)^k \dfrac{\beta^{2k}}{(2k)!} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$ ■

Example. Now let

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda \neq 0$. With an eye toward what comes later, we compute, not $\exp A$, but rather $\exp(tA)$. We have

$$(tA)^k = \begin{pmatrix} (t\lambda)^k & kt^k\lambda^{k-1} \\ 0 & (t\lambda)^k \end{pmatrix}.$$

Hence we find

$$\exp(tA) = \begin{pmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} & \displaystyle t\sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} \\ 0 & \displaystyle\sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}.$$

■

Note that, in each of these three examples, the matrix $\exp(A)$ is a matrix whose entries are infinite series. We therefore say that the infinite series of matrices $\exp(A)$ converges absolutely if each of its individual terms does so. In each of the previous cases, this convergence was clear. Unfortunately, in the case of a general matrix A, this is not so clear. To prove convergence here, we need to work a little harder.

Let $a_{ij}(k)$ denote the ij entry of A^k. Let $a = \max |a_{ij}|$. We have

$$|a_{ij}(2)| = \left| \sum_{k=1}^{n} a_{ik}a_{kj} \right| \leq na^2$$

$$|a_{ij}(3)| = \left| \sum_{k,\ell=1}^{n} a_{ik}a_{k\ell}a_{\ell j} \right| \leq n^2a^3$$

$$\vdots$$

$$|a_{ij}(k)| \leq n^{k-1}a^k.$$

Hence we have a bound for the ij entry of the $n \times n$ matrix $\exp(A)$:

$$\left| \sum_{k=0}^{\infty} \frac{a_{ij}(k)}{k!} \right| \leq \sum_{k=0}^{\infty} \frac{|a_{ij}(k)|}{k!} \leq \sum_{k=0}^{\infty} \frac{n^{k-1}a^k}{k!} \leq \sum_{k=0}^{\infty} \frac{(na)^k}{k!} \leq \exp na$$

so that this series converges absolutely by the comparison test. Therefore the matrix $\exp A$ makes sense for any $A \in L(\mathbb{R}^n)$.

The following result shows that matrix exponentiation shares many of the familiar properties of the usual exponential function.

Proposition. *Let A, B, and T be $n \times n$ matrices. Then:*

1. *If* $B = T^{-1}AT$, *then* $\exp(B) = T^{-1}\exp(A)T$.
2. *If* $AB = BA$, *then* $\exp(A+B) = \exp(A)\exp(B)$.
3. $\exp(-A) = (\exp(A))^{-1}$.

Proof: The proof of (1) follows from the identities $T^{-1}(A+B)T = T^{-1}AT + T^{-1}BT$ and $(T^{-1}AT)^k = T^{-1}A^kT$. Therefore

$$T^{-1}\left(\sum_{k=0}^{n}\frac{A^k}{k!}\right)T = \sum_{k=0}^{n}\frac{\left(T^{-1}AT\right)^k}{k!}$$

and (1) follows by taking limits.

To prove (2), observe that because $AB = BA$ we have, by the binomial theorem,

$$(A+B)^n = n!\sum_{j+k=n}\frac{A^j}{j!}\frac{B^k}{k!}.$$

Therefore we must show that

$$\sum_{n=0}^{\infty}\left(\sum_{j+k=n}\frac{A^j}{j!}\frac{B^k}{k!}\right) = \left(\sum_{j=0}^{\infty}\frac{A^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right).$$

This is not as obvious as it may seem, since we are dealing here with series of matrices, not series of real numbers. So we will prove this in the following lemma, which then proves (2). Putting $B = -A$ in (2) gives (3). ■

Lemma. *For any $n \times n$ matrices A and B, we have:*

$$\sum_{n=0}^{\infty}\left(\sum_{j+k=n}\frac{A^j}{j!}\frac{B^k}{k!}\right) = \left(\sum_{j=0}^{\infty}\frac{A^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right).$$

Proof: We know that each of these infinite series of matrices converges. We just have to check that they converge to each other. To do this, consider the partial sums

$$\gamma_{2m} = \sum_{n=0}^{2m}\left(\sum_{j+k=n}\frac{A^j}{j!}\frac{B^k}{k!}\right)$$

and

$$\alpha_m = \left(\sum_{j=0}^{m} \frac{A^j}{j!}\right) \text{ and } \beta_m = \left(\sum_{k=0}^{m} \frac{B^k}{k!}\right).$$

We need to show that the matrices $\gamma_{2m} - \alpha_m\beta_m$ tend to the zero matrix as $m \to \infty$. Toward that end, for a matrix $M = [m_{ij}]$, we let $\|M\| = \max |m_{ij}|$. We will show that $\|\gamma_{2m} - \alpha_m\beta_m\| \to 0$ as $m \to \infty$.

A computation shows that

$$\gamma_{2m} - \alpha_m\beta_m = \sum{}' \frac{A^j}{j!}\frac{B^k}{k!} + \sum{}'' \frac{A^j}{j!}\frac{B^k}{k!}$$

where $\sum'$ denotes the sum over terms with indices satisfying

$$j+k \le 2m, \quad 0 \le j \le m, \quad m+1 \le k \le 2m$$

while $\sum''$ is the sum corresponding to

$$j+k \le 2m, \quad m+1 \le j \le 2m, \quad 0 \le k \le m.$$

Therefore

$$\|\gamma_{2m} - \alpha_m\beta_m\| \le \sum{}' \left\|\frac{A^j}{j!}\right\| \cdot \left\|\frac{B^k}{k!}\right\| + \sum{}'' \left\|\frac{A^j}{j!}\right\| \cdot \left\|\frac{B^k}{k!}\right\|.$$

Now

$$\sum{}' \left\|\frac{A^j}{j!}\right\| \cdot \left\|\frac{B^k}{k!}\right\| \le \left(\sum_{j=0}^{m} \left\|\frac{A^j}{j!}\right\|\right)\left(\sum_{k=m+1}^{2m} \left\|\frac{B^k}{k!}\right\|\right).$$

This tends to 0 as $m \to \infty$ since, as we saw above,

$$\sum_{j=0}^{\infty} \left\|\frac{A^j}{j!}\right\| \le \exp(n\|A\|) < \infty.$$

Similarly,

$$\sum{}'' \left\|\frac{A^j}{j!}\right\| \cdot \left\|\frac{B^k}{k!}\right\| \to 0$$

as $m \to \infty$. Therefore $\lim_{m\to\infty}(\gamma_{2m} - \alpha_m\beta_m) = 0$, proving the lemma. ■

Observe that statement (3) of the proposition implies that $\exp(A)$ is invertible for every matrix A. This is analogous to the fact that $e^a \neq 0$ for every real number a.

There is a very simple relationship between the eigenvectors of A and those of $\exp(A)$.

Proposition. *If $V \in \mathbb{R}^n$ is an eigenvector of A associated to the eigenvalue λ, then V is also an eigenvector of $\exp(A)$ associated to e^λ.*

Proof: From $AV = \lambda V$, we obtain

$$\begin{aligned}
\exp(A)V &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{A^k V}{k!} \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{\lambda^k}{k!} V \right) \\
&= \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) V \\
&= e^\lambda V.
\end{aligned}$$

■

Now let's return to the setting of systems of differential equations. Let A be an $n \times n$ matrix and consider the system $X' = AX$. Recall that $L(\mathbb{R}^n)$ denotes the set of all $n \times n$ matrices. We have a function $\mathbb{R} \to L(\mathbb{R}^n)$, which assigns the matrix $\exp(tA)$ to $t \in \mathbb{R}$. Since $L(\mathbb{R}^n)$ is identified with $\mathbb{R}^{n^2}$, it makes sense to speak of the derivative of this function.

Proposition.

$$\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A.$$

In other words, the derivative of the matrix-valued function $t \to \exp(tA)$ is another matrix-valued function $A\exp(tA)$.

Proof: We have

$$\begin{aligned}
\frac{d}{dt}\exp(tA) &= \lim_{h\to 0} \frac{\exp((t+h)A) - \exp(tA)}{h} \\
&= \lim_{h\to 0} \frac{\exp(tA)\exp(hA) - \exp(tA)}{h}
\end{aligned}$$

$$
\begin{aligned}
&= \exp(tA) \lim_{h\to 0} \left(\frac{\exp(hA) - I}{h} \right) \\
&= \exp(tA)A;
\end{aligned}
$$

that the last limit equals A follows from the series definition of $\exp(hA)$. Note that A commutes with each term of the series for $\exp(tA)$, hence with $\exp(tA)$. This proves the proposition. ■

Now we return to solving systems of differential equations. The following may be considered as the fundamental theorem of linear differential equations with constant coefficients.

Theorem. *Let A be an $n \times n$ matrix. Then the solution of the initial value problem $X' = AX$ with $X(0) = X_0$ is $X(t) = \exp(tA)X_0$. Moreover, this is the only such solution.*

Proof: The preceding proposition shows that

$$
\frac{d}{dt}(\exp(tA)X_0) = \left(\frac{d}{dt} \exp(tA) \right) X_0 = A \exp(tA)X_0.
$$

Moreover, since $\exp(0A)X_0 = X_0$, it follows that this is a solution of the initial value problem. To see that there are no other solutions, let $Y(t)$ be another solution satisfying $Y(0) = X_0$ and set

$$
Z(t) = \exp(-tA)Y(t).
$$

Then

$$
\begin{aligned}
Z'(t) &= \left(\frac{d}{dt} \exp(-tA) \right) Y(t) + \exp(-tA)Y'(t) \\
&= -A \exp(-tA)Y(t) + \exp(-tA)AY(t) \\
&= \exp(-tA)(-A + A)Y(t) \\
&\equiv 0.
\end{aligned}
$$

Therefore $Z(t)$ is a constant. Setting $t = 0$ shows $Z(t) = X_0$, so that $Y(t) = \exp(tA)X_0$. This completes the proof of the theorem. ■

Note that this proof is identical to that given way back in Section 1.1. Only the meaning of the letter A has changed.

Example. Consider the system

$$X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X.$$

By the theorem, the general solution is

$$X(t) = \exp(tA)X_0 = \exp \begin{pmatrix} t\lambda & t \\ 0 & t\lambda \end{pmatrix} X_0.$$

But this is precisely the matrix whose exponential we computed earlier. We find

$$X(t) = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix} X_0.$$

Note that this agrees with our computations in Chapter 3. ■

6.5 Nonautonomous Linear Systems

Up to this point, virtually all of the linear systems of differential equations that we have encountered have been autonomous. There are, however, certain types of nonautonomous systems that often arise in applications. One such system is of the form

$$X' = A(t)X$$

where $A(t) = [a_{ij}(t)]$ is an $n \times n$ matrix that depends continuously on time. We will investigate these types of systems further when we encounter the variational equation in subsequent chapters.

Here we restrict our attention to a different type of nonautonomous linear system given by

$$X' = AX + G(t)$$

where A is a constant $n \times n$ matrix and $G : \mathbb{R} \to \mathbb{R}^n$ is a *forcing term* that depends explicitly on t. This is an example of a first-order, linear, nonhomogeneous system of equations.

Example. (Forced Harmonic Oscillator) If we apply an external force to the harmonic oscillator system, the differential equation governing the motion becomes

$$x'' + bx' + kx = f(t)$$

where $f(t)$ measures the external force. An important special case occurs when this force is a periodic function of time, which corresponds, for example, to moving the table on which the mass-spring apparatus resides back and forth periodically. As a system, the forced harmonic oscillator equation becomes

$$X' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} X + G(t) \quad \text{where} \quad G(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \qquad \blacksquare$$

For a nonhomogeneous system, the equation that results from dropping the time-dependent term, namely, $X' = AX$, is called the *homogeneous equation.* We know how to find the general solution of this system. Borrowing the notation from the previous section, the solution satisfying the initial condition $X(0) = X_0$ is

$$X(t) = \exp(tA)X_0,$$

so this is the general solution of the homogeneous equation.

To find the general solution of the nonhomogeneous equation, suppose that we have one particular solution $Z(t)$ of this equation. So $Z'(t) = AZ(t) + G(t)$. If $X(t)$ is any solution of the homogeneous equation, then the function $Y(t) = X(t) + Z(t)$ is another solution of the nonhomogeneous equation. This follows since we have

$$\begin{aligned} Y' &= X' + Z' = AX + AZ + G(t) \\ &= A(X + Z) + G(t) \\ &= AY + G(t). \end{aligned}$$

Therefore, since we know all solutions of the homogeneous equation, we can now find the general solution to the nonhomogeneous equation, provided that we can find just one particular solution of this equation. Often one gets such a solution by simply guessing that solution (in calculus, this method is usually called the *method of undetermined coefficients*). Unfortunately, guessing a solution does not always work. The following method, called *variation of parameters*, does work in all cases. However, there is no guarantee that we can actually evaluate the required integrals.

Theorem. (Variation of Parameters) *Consider the nonhomogeneous equation*

$$X' = AX + G(t)$$

where A is an $n \times n$ matrix and $G(t)$ is a continuous function of t. Then

$$X(t) = \exp(tA)\left(X_0 + \int_0^t \exp(-sA)\, G(s)\, ds\right)$$

is a solution of this equation satisfying $X(0) = X_0$.

Proof: Differentiating $X(t)$, we obtain

$$\begin{aligned} X'(t) &= A\exp(tA)\left(X_0 + \int_0^t \exp(-sA)\, G(s)\, ds\right) \\ &\quad + \exp(tA)\,\frac{d}{dt}\int_0^t \exp(-sA)\, G(s)\, ds \\ &= A\exp(tA)\left(X_0 + \int_0^t \exp(-sA)\, G(s)\, ds\right) + G(t) \\ &= AX(t) + G(t). \end{aligned}$$

■

We now give several applications of this result in the case of the periodically forced harmonic oscillator. Assume first that we have a damped oscillator that is forced by $\cos t$, so the period of the forcing term is 2π. The system is

$$X' = AX + G(t)$$

where $G(t) = (0, \cos t)$ and A is the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}$$

with $b, k > 0$. We claim that there is a unique periodic solution of this system that has period 2π. To prove this, we must first find a solution $X(t)$ satisfying $X(0) = X_0 = X(2\pi)$. By variation of parameters, we need to find X_0 such that

$$X_0 = \exp(2\pi A)X_0 + \exp(2\pi A)\int_0^{2\pi} \exp(-sA)\, G(s)\, ds.$$

Now the term

$$\exp(2\pi A)\int_0^{2\pi} \exp(-sA)\, G(s)\, ds$$

is a constant vector, which we denote by W. Therefore we must solve the equation

$$\big(\exp(2\pi A) - I\big)\, X_0 = -W.$$

There is a unique solution to this equation, since the matrix $\exp(2\pi A) - I$ is invertible. For if this matrix were not invertible, we would have a nonzero vector V with

$$\big(\exp(2\pi A) - I\big)\, V = 0,$$

or, in other words, the matrix $\exp(2\pi A)$ would have an eigenvalue of 1. But, from the previous section, the eigenvalues of $\exp(2\pi A)$ are given by $\exp(2\pi \lambda_j)$ where the λ_j are the eigenvalues of A. But each λ_j has real part less than 0, so the magnitude of $\exp(2\pi\lambda_j)$ is smaller than 1. Thus the matrix $\exp(2\pi A) - I$ is indeed invertible, and the unique initial value leading to a 2π-periodic solution is

$$X_0 = \big(\exp(2\pi A) - I\big)^{-1}\,(-W).$$

So let $X(t)$ be this periodic solution with $X(0) = X_0$. This solution is called the *steady-state* solution. If Y_0 is any other initial condition, then we may write $Y_0 = (Y_0 - X_0) + X_0$, so the solution through Y_0 is given by

$$\begin{aligned} Y(t) &= \exp(tA)(Y_0 - X_0) + \exp(tA)X_0 + \exp(tA)\int_0^t \exp(-sA)\, G(s)\, ds \\ &= \exp(tA)(Y_0 - X_0) + X(t). \end{aligned}$$

The first term in this expression tends to 0 as $t \to \infty$, since it is a solution of the homogeneous equation. Hence every solution of this system tends to the steady-state solution as $t \to \infty$. Physically, this is clear: The motion of the damped (and unforced) oscillator tends to equilibrium, leaving only the motion due to the periodic forcing. We have proved:

Theorem. *Consider the forced, damped harmonic oscillator equation*

$$x'' + bx' + kx = \cos t$$

with $k, b > 0$. Then all solutions of this equation tend to the steady-state solution, which is periodic with period 2π. ■

Now consider a particular example of a forced, undamped harmonic oscillator

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ \cos \omega t \end{pmatrix}$$

where the period of the forcing is now $2\pi/\omega$ with $\omega \neq \pm 1$. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The solution of the homogeneous equation is

$$X(t) = \exp(tA)X_0 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} X_0.$$

Variation of parameters provides a solution of the nonhomogeneous equation starting at the origin:

$$\begin{aligned} Y(t) &= \exp(tA) \int_0^t \exp(-sA) \begin{pmatrix} 0 \\ \cos \omega s \end{pmatrix} ds \\ &= \exp(tA) \int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ \cos \omega s \end{pmatrix} ds \\ &= \exp(tA) \int_0^t \begin{pmatrix} -\sin s \cos \omega s \\ \cos s \cos \omega s \end{pmatrix} ds \\ &= \frac{1}{2} \exp(tA) \int_0^t \begin{pmatrix} \sin(\omega - 1)s - \sin(\omega + 1)s \\ \cos(\omega - 1)s + \cos(\omega + 1)s \end{pmatrix} ds. \end{aligned}$$

Recalling that

$$\exp(tA) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and using the fact that $\omega \neq \pm 1$, evaluation of this integral plus a long computation yields

$$Y(t) = \frac{1}{2}\exp(tA)\begin{pmatrix} \dfrac{-\cos(\omega-1)t}{\omega-1} + \dfrac{\cos(\omega+1)t}{\omega+1} \\ \dfrac{\sin(\omega-1)t}{\omega-1} + \dfrac{\sin(\omega+1)t}{\omega+1} \end{pmatrix}$$

$$+\exp(tA)\begin{pmatrix} (\omega^2-1)^{-1} \\ 0 \end{pmatrix}$$

$$= \frac{1}{\omega^2-1}\begin{pmatrix} -\cos\omega t \\ \omega\sin\omega t \end{pmatrix} + \exp(tA)\begin{pmatrix} (\omega^2-1)^{-1} \\ 0 \end{pmatrix}.$$

Thus the general solution of this equation is

$$Y(t) = \exp(tA)\left(X_0 + \begin{pmatrix} (\omega^2-1)^{-1} \\ 0 \end{pmatrix}\right) + \frac{1}{\omega^2-1}\begin{pmatrix} -\cos\omega t \\ \omega\sin\omega t \end{pmatrix}.$$

The first term in this expression is periodic with period 2π while the second has period $2\pi/\omega$. Unlike the damped case, this solution does not necessarily yield a periodic motion. Indeed, this solution is periodic if and only if ω is a rational number. If ω is irrational, the motion is quasiperiodic, just as we saw in Section 6.2.

EXERCISES

1. Find the general solution for $X' = AX$ where A is given by:

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

2. Consider the linear system

$$X' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} X$$

where $\lambda_3 < \lambda_2 < \lambda_1 < 0$. Describe how the solution through an arbitrary initial value tends to the origin.

3. Give an example of a 3×3 matrix A for which all nonequilibrium solutions of $X' = AX$ are periodic with period 2π. Sketch the phase portrait.

4. Find the general solution of

$$X' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} X.$$

5. Consider the system

$$X' = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix} X$$

depending on the two parameters a and b.

(a) Find the general solution of this system.

(b) Sketch the region in the ab–plane where this system has different types of phase portraits.

6. Consider the system

$$X' = \begin{pmatrix} a & 0 & b \\ 0 & b & 0 \\ -b & 0 & a \end{pmatrix} X$$

depending on the two parameters a and b.

(a) Find the general solution of this system.

(b) Sketch the region in the ab–plane where this system has different types of phase portraits.

7. **Coupled Harmonic Oscillators.** In this series of exercises you are asked to generalize the material on harmonic oscillators in Section 6.2 to the case where the oscillators are *coupled.* Suppose there are two masses m_1

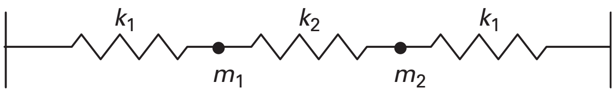


Figure 6.10 A coupled oscillator.

and m_2 attached to springs and walls as shown in Figure 6.10. The springs connecting m_j to the walls both have spring constants k_1, while the spring connecting m_1 and m_2 has spring constant k_2. This coupling means that the motion of either mass affects the behavior of the other.

Let x_j denote the displacement of each mass from its rest position, and assume that both masses are equal to 1. The differential equations for these coupled oscillators are then given by

$$\begin{aligned} x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 \\ x_2'' &= k_2 x_1 - (k_1 + k_2)x_2. \end{aligned}$$

These equations are derived as follows. If m_1 is moved to the right ($x_1 > 0$), the left spring is stretched and exerts a restorative force on m_1 given by $-k_1 x_1$. Meanwhile, the central spring is compressed, so it exerts a restorative force on m_1 given by $-k_2 x_1$. If the right spring is stretched, then the central spring is compressed and exerts a restorative force on m_1 given by $k_2 x_2$ (since $x_2 < 0$). The forces on m_2 are similar.

(a) Write these equations as a first-order linear system.
(b) Determine the eigenvalues and eigenvectors of the corresponding matrix.
(c) Find the general solution.
(d) Let $\omega_1 = \sqrt{k_1}$ and $\omega_2 = \sqrt{k_1 + 2k_2}$. What can be said about the periodicity of solutions relative to the ω_j? Prove this.

8. Suppose $X' = AX$ where A is a 4×4 matrix whose eigenvalues are $\pm i\sqrt{2}$ and $\pm i\sqrt{3}$. Describe this flow.

9. Suppose $X' = AX$ where A is a 4×4 matrix whose eigenvalues are $\pm i$ and $-1 \pm i$. Describe this flow.

10. Suppose $X' = AX$ where A is a 4×4 matrix whose eigenvalues are $\pm i$ and ± 1. Describe this flow.

11. Consider the system $X' = AX$ where $X = (x_1, \ldots, x_6)$ and

$$A = \begin{pmatrix} 0 & \omega_1 & & & & \\ -\omega_1 & 0 & & & & \\ & & 0 & \omega_2 & & \\ & & -\omega_2 & 0 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix}$$

and ω_1/ω_2 is irrational. Describe qualitatively how a solution behaves when, at time 0, each x_j is nonzero with the exception that

(a) $x_6 = 0$;
(b) $x_5 = 0$;
(c) $x_3 = x_4 = x_5 = 0$;
(d) $x_3 = x_4 = x_5 = x_6 = 0$.

12. Compute the exponentials of the following matrices:

(a) $\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ (g) $\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$

(h) $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ (i) $\begin{pmatrix} 1+i & 0 \\ 2 & 1+i \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

13. Find an example of two matrices A, B such that

$$\exp(A + B) \neq \exp(A)\exp(B).$$

14. Show that if $AB = BA$, then

(a) $\exp(A)\exp(B) = \exp(B)\exp(A)$
(b) $\exp(A)B = B\exp(A)$.

15. Consider the triplet of harmonic oscillators

$$\begin{aligned} x_1'' &= -x_1 \\ x_2'' &= -2x_2 \\ x_3'' &= -\omega^2 x_3 \end{aligned}$$

where ω is irrational. What can you say about the qualitative behavior of solutions of this six-dimensional system?